LENGTH DISTORTION OF CURVES UNDER CONFORMAL MAPPINGS

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1. INTRODUCTION

Let α denote the open upper half of the unit circle C: $|z| = 1$, let α^* denote the open real diameter of the unit disk K: $|z| < 1$, and let us consider a conformal mapping f of K onto a simply connected domain D in the finite plane. The image $\beta^* = f\alpha^*$ of α^* is a locally rectifiable curve with length $\|\beta^*\| \leq \infty$, and α corresponds to a "curve" β on the boundary of D to which we can assign a "length" $\|\beta\| \leq \infty$.

An unpublished but widely circulated conjecture by Piranian states that there exists a *finite constant* A such that $\|\beta^*\| \leq A \cdot \|\beta\|$, and that the *best possible value* A_0 of the constant is π. Gehring and Hayman [1, Theorem 1] proved the first part of the conjecture, and they showed that $\pi \leq A_0 < 74$.

In Section 7, we disprove the second part of Piranian's conjecture: by means of an example, we show that $A_0 \geq 4.5$. In Section 6, we reduce the upper estimate of A_0 to 17.5. Since the first part of the conjecture can not be extended to quasiconformal mappings, the proof for the upper estimate must involve conformality in an essential way; indeed, we use the distortion $|dw|/|dz| = |f'(z^*)|$ under the mapping $w = f(z)$ at the points z^* of α^* in order to get the length $\|\beta^*\|$.

We shall consider all circular arcs α^* in K on which the harmonic measure of α has the constant value ω; the original problem is the special case $\omega = 1/2$. In Sections 3 to 5, we give certain lower estimates for the length of β; they depend on the harmonic measure ω of β at an interior point $w^* = f(z^*)$ of D, and they are either proportional to the distance of w^* from β or proportional to the distortion $|f'(z^*)|$ at the point z^*.

In Section 2 we discuss "curves" on the boundary of an arbitrary simply connected domain. Without this generality, we should repeatedly be forced to put clumsy restrictions on the domains D and on the conformal mappings f to be admitted.

2. CURVES ON THE BOUNDARY

We consider a simply connected domain D in the extended complex plane, the *abstract boundary* ∂D consisting of the *prime ends* $\mathscr{W}$, and the *projections* $W = p\mathscr{W}$ into the plane. We define a *semidistance* $\rho(\mathscr{W}_1, \mathscr{W}_2)$ for prime ends as the infimum of those constants r such that for each point of $p\mathscr{W}_1$ and of $p\mathscr{W}_2$ there is a point of the other set within euclidean distance r.

If f denotes a conformal mapping of K onto D, we shall use the same symbol f for the mapping that carries the points of C onto the corresponding prime ends of D. The induced *cyclic ordering* of ∂D allows us to speak of *intervals* on the abstract boundary. We regard an open interval β on ∂D as a generalized *curve,* because it is the image $f\alpha$ under f of an interval α on the unit circle C.

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