ON BIORTHOGONAL SYSTEMS

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1. INTRODUCTION

Let H be a separable Hilbert space. A sequence of pairs $\{x_n, y_n\}$ of elements of H is said to be a *biorthonormal system* if

$$(x_n, y_m) = \delta_{nm} = \begin{cases} 1 & (n = m), \\ 0 & (n \neq m). \end{cases}$$

A biorthonormal system $\{x_n, y_n\}$ is said to be *complete* if every $f \in H$ can be written in the form

$$f = \sum_{n=1}^{\infty} (f, y_n) x_n = \sum_{n=1}^{\infty} (f, x_n) y_n.$$

A sequence $\{\phi_n\}$ of elements of H is said to be *orthonormal* if the system $\{\phi_n, \phi_n\}$ is a biorthonormal system. An orthonormal sequence $\{\phi_n\}$ is said to be *complete* if the system $\{\phi_n, \phi_n\}$ is complete.

Concerning the relation between the completeness of biorthonormal systems and the asymptotic estimates for the eigenfunctions of a Sturm-Liouville problem, F. Brauer [2], [3] proved the following theorem.

THEOREM 1. *Let* $\{\phi_n\}$ *be a complete orthonormal sequence, and let* $\{x_n, y_n\}$ *be a biorthonormal system such that*

$$\sum_{n=1}^{\infty} \|\phi_n - x_n\|^2 < +\infty, \qquad \sum_{n=1}^{\infty} \|\phi_n - y_n\|^2 < +\infty.$$

Then the system $\{x_n, y_n\}$ *is complete.*

Let us define a linear transformation K of H into itself by $x_n - \phi_n = K\phi_n$ $(n = 1, 2, \cdots)$. Under the hypotheses of Theorem 1, it can be shown that K is a bounded transformation (F. Brauer [2, p. 380]). In order to prove Theorem 1, it is sufficient to prove the existence of the bounded inverse transformation of $I + K$, where I is the identity transformation. In particular, if the norm of K is less than 1, the bounded inverse transformation of $I + K$ is given by the Neumann series $\sum_{m=0}^{\infty}(-K)^m$. This is the essential part of the Riesz-Nagy proof of the Paley-Wiener theorem on perturbations of orthonormal sequences (F. Riesz and B. Sz.-Nagy [4, pp. 208-209]). Notice that, if the norm of K is less than 1, we do not need to assume anything on $\{y_n\}$. Actually, the Paley-Wiener theorem can be stated as follows:

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