REMARK ON A RESULT OF KAPLANSKY CONCERNING C(X)

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In this paper, let R be a chain (with more than one element) endowed with the interval topology, let X be a compact Hausdorff space, and let $R(X)$ denote the family of continuous functions in R^X. For $f, g \in R(X)$, let $f \leq g$ mean that $f(x) \leq g(x)$ for all $x \in X$, and let $f < g$ mean that $f(x) < g(x)$ for all $x \in X$. For $f \in R(X)$, let G_f denote the graph of f in $X \times R$. Under the partial ordering $\leq$, $R(X)$ is a lattice.

In [4] Kaplansky described all the lattice automorphisms of $R(X)$ that are bicontinuous in the topology of uniform convergence (here R is the real line); if ϕ is such an automorphism, there exist a homeomorphism T of X onto X and a continuous mapping p of $X \times R$ into R such that $\phi(f)(Tx) = p(x, f(x))$ for all $x \in X$ and all $f \in R(X)$, and for each $x \in X$ the mapping $r \to p(x, r)$ is increasing. (Milgram [5] presents a similar result for the case where $R(X)$ is regarded as a multiplicative semigroup.) He observed that if X satisfies the first countability axiom, then each automorphism ϕ of $R(X)$ must be bicontinuous, and hence of this form. Finally, he presented a compact space X and an automorphism ϕ of $R(X)$ that cannot be so described [4, p. 629].

We shall present analogues of these results in a much broader context in which Kaplansky's arguments do not apply (see Examples 1 and 2). The prime ideals employed in [3] and [4] will not enter our development of Theorems I, II, and III.

Definition 1. A sublattice L of $R(X)$ is an R-sublattice if (1) for each $x \in X$, $Lx = \{f(x)\colon f \in L\}$ consists of more than one element, and (2) given $f_1, f_2 \in L$, $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists an $h \in L$ such that $h(x_i) = f_i(x_i)$ for $i = 1, 2$.

Note that a characterizing sublattice of $R(X)$ in the sense of Anderson and Blair [1] is an R-sublattice. For if

$$h_1(x_1) < f_1(x_1), \qquad h_1(x_2) > f_2(x_2), \qquad h_2(x_1) > f_1(x_1), \qquad h_2(x_2) < f_2(x_2),$$

then $(h_1 \vee f_1) \wedge (h_2 \vee f_2)$ coincides with f_1 at x_1 and with f_2 at x_2. On the other hand, an R-sublattice L is characterizing if and only if for each $x \in X$, Lx has no maximal or minimal element. (See Examples 1 and 2, and also compare R-sublattices with the c-characterizing lattices of Blair [2].)

Throughout this paper, L_1 and L_2 will be R-sublattices of $R(X)$, and ϕ will be a lattice isomorphism of L_1 onto L_2.

Definition 2. The isomorphism ϕ of L_1 onto L_2 is increasing if for $f, g \in L_1$, $f < g$ if and only if $\phi(f) < \phi(g)$.

THEOREM I. *A necessary and sufficient condition that the isomorphism* ϕ *of* L_1 *onto* L_2 *be increasing is that there exist a homeomorphism* T *of* X *onto* X *and a mapping* p *of* $\bigcup_{f \in L_1} G_f$ *into* R, *continuous on each* G_f, *such that for each* $x \in X$, $r \to p(x, r)$ *is an increasing mapping of* $L_1 x$ *onto* $L_2(Tx)$, *and such that*

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