THEOREMS ON BREWER AND JACOBSTHAL SUMS. II

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1. INTRODUCTION

Let $V_n(x)$ be the polynomial determined by the recurrence relation

$$V_{n+2}(x) = xV_{n+1}(x) - V_n(x) \qquad (n = 1, 2, \cdots)$$

with $V_1(x) = x$, $V_2(x) = x^2 - 2$. In a recent paper [1], B. W. Brewer has defined the sum

$$\Lambda_n = \sum_{s=0}^{p-1} \chi(V_n(s)),$$

where $\chi(s)$ denotes the Legendre symbol (s/p) and p is an odd prime. It is easily verified that $\Lambda_1 = 0$, $\Lambda_2 = -1$. Brewer evaluated Λ_3, Λ_4, and Λ_5. For a summary of the results pertaining to Λ_3 and Λ_4, see Part I of the present paper [9].

The results for the sum $\Lambda_5 = \Sigma_{s=0}^{p-1} \chi(s(s^4 - 5s^2 + 5))$ are as follows.

If $p \equiv 3 \pmod 4$ or if $p \equiv \pm 2 \pmod 5$, then $\Lambda_5 = 0$.

If $p = 20f + 1 = u^2 + 5v^2 = x^2 + 4y^2$ with $x \equiv 1 \pmod 4$, then $\Lambda_5 = 0$ if $5 \mid x$, and $\Lambda_5 = -4u$ if $5 \nmid x$ and $u \equiv x \pmod 5$.

If $p = 20f + 9 = u^2 + 5v^2 = x^2 + 4y^2$ with $x \equiv 1 \pmod 4$, then $\Lambda_5 = 0$ if $5 \mid x$, and $\Lambda_5 = 4u$ if $5 \nmid x$ and $u \equiv x \pmod 5$.

Moreover, the following congruences modulo p hold:

$$(1.1) \qquad \binom{10f}{f}\binom{10f}{3f} \equiv 4u^2, \qquad \binom{10f}{f} \equiv \pm\binom{10f}{3f} \qquad (p = 20f + 1),$$

$$(1.2) \qquad \binom{10f+4}{f}\binom{10f+4}{3f+1} \equiv 4u^2, \qquad \binom{10f+4}{f} \equiv \pm\binom{10f+4}{3f+1} \qquad (p = 20f + 9).$$

Brewer bases his method for evaluating Λ_5 upon the congruences in (1.1) and (1.2). The purpose of the present paper is to derive the results for Λ_5 without employing these congruences. In Part I, which has been published elsewhere [9], the following theorems are established. Theorem 1 gives the value of Λ_5 when $p \equiv \pm 2 \pmod 5$. Theorem 2 gives the value of Λ_5 when $p = 20f + 1$. Theorem 3 is a statement of the two congruences in (1.1) together with a resolution of the ambiguous sign in the second congruence. Let $p = 20f + 1$ and put $p = x^2 + 4y^2$. Theorem 3 asserts that then the ambiguous sign is plus if $5 \nmid x$ and is minus if $5 \mid x$.

The theory of cyclotomy modulo a prime $p = ef + 1$ leads, for $e = 20$, to the case $p = 20f + 1$. The method of [9] is based on this theory and was suggested by Cauchy's

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