NOTE ON AN INVARIANT OF KERVAIRE

Edgar H. Brown, Jr.

In [2] Kervaire defined the so-called Arf invariant $\Phi(M) \in Z_2$ for an $(n-1)$-connected, compact, closed, C^∞ manifold M of dimension 2n, where n is odd and $n \neq 1, 3, 7$ (see also [3]). In fact, he showed that Φ induces a homomorphism from the $2n^{\text{th}}$ framed cobordism group into Z_2, and that $\Phi = 0$ if $n = 5$. It is an unsolved problem whether $\Phi = 0$ for all n.

Let $\Omega_m(\text{Spin})$ denote the m^{th} spin or cobordism group [4]. The aim of this note is to generalize Φ in the following sense. We define a homomorphism

$$\Psi: \Omega_{2n}(\text{Spin}) \to Z_2$$

for $n \equiv 1 \pmod 4$ such that $\Psi(M) = \Phi(M)$ if M is as above. The writer has not been able to show that $\Psi \neq 0$. It is known that the image of framed cobordism in Spin cobordism is not zero. (Milnor has shown that there exists a homotopy 10-sphere that is not a Spin boundary.)

In the following, $n \equiv 1 \pmod 4$, $n > 1$, and all cohomology groups have Z_2 coefficients. Recall that

$$Sq^{n+1} = Sq^2 Sq^{n-1} + Sq^1 Sq^2 Sq^{n-2}.$$

Hence, on n-dimensional cohomology classes,

$$Sq^2 Sq^{n-1} + Sq^1 (Sq^2 Sq^{n-2})$$

is a relation. In [1] it is shown that such a relation gives rise to a secondary cohomology operation

$$\psi: H^n(X) \cap \text{Ker } Sq^{n-1} \cap \text{Ker } Sq^2 Sq^{n-2} \to H^{2n}(X)/Sq^2 H^{2n-2}(X) + Sq^1 H^{2n-1}(X).$$

Furthermore, if $\psi(u)$ and $\psi(v)$ are defined, then $\psi(u+v)$ is defined and

$$\psi(u+v) = \psi(u) + \psi(v) + u \cup v$$

modulo the indeterminacy of the operation.

Suppose M is a closed, compact, simply connected 2n-manifold such that the Stiefel-Whitney class $W_2(M)$ is zero. If $u \in H^n(M)$, then

$$Sq^{n-1} u \in H^{2n-1}(M) = 0, \qquad Sq^2 Sq^{n-2} u = W_2 Sq^{n-2} u = 0,$$

$$Sq^2 H^{2n-2}(M) = W_2 H^{2n-2}(M) = 0, \qquad \text{and} \qquad Sq^1 H^{2n-1}(M) = 0.$$

Hence ψ defines a quadradic function

$$\psi: H^n(M) \to H^{2n}(M).$$

Received July 2, 1964.