CONNEXION PRESERVING, CONFORMAL, AND PARALLEL MAPS

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This paper is a small collection of loosely related results in differential geometry. The methods are perhaps more interesting than the results for they illustrate the power and elegance of completely invariant methods in differential geometry.

Preliminaries. Let M and M' be C^∞ Riemannian manifolds (where we denote the metric tensor by $\langle X, Y\rangle$), and let f: $M \to M'$ be a C^∞ map. If there exists a C^∞ real-valued function F on M such that for any m in M, then

$$\langle f_* X, f_* Y\rangle = F(m)\langle X, Y\rangle$$

for all X, Y in M_m; and if $F > 0$ on M, then f is *conformal.* The map f_* is the differential of f, and f_* has no kernel if f is conformal. We call the map F the *scale function;* notice $F \geq 0$. If F is a constant function, then we say f is *homothetic.* If F = 1, we say f is an *isometry,* and if f is both an isometry and a diffeomorphism, we say M is *isometric* to M'.

A *connexion* D on M will be a C^∞ covariant differentiation operator assigning to C^∞ fields X and Y (with common domain A) a C^∞ field $D_X Y$ (on A) such that

$$D_{(X+Z)}Y = D_X Y + D_Z Y ,$$

$$D_X(Z + Y) = D_X Z + D_X Y ,$$

$$D_{fX} Y = fD_X Y ,$$

$$D_X fY = (Xf)Y + fD_X Y ,$$

where Z is a C^∞ field on A and f is a C^∞ real valued function on A. The *torsion* tensor T(X, Y) and *curvature* tensor R(X, Y) of a connexion are defined by

$$T(X, Y) = D_X Y - D_Y X - [X, Y]$$

$$R(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]} .$$

We will use the fact that on a Riemannian manifold there exists a unique (Riemannian) connexion D with zero torsion which satisfies the property

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle ,$$

where Y and Z are vector fields in a neighborhood of the base point of the vector X. If M and M' are C^∞ manifolds with connexions D and D', respectively, then a map f: $M \to M'$ is *connexion preserving* if $f_* D_X Y = D'_{f_* X} f_* Y$ for all vectors X and vector fields Y. (Note $D'_{f_* X} f_* Y$ is well defined since $f_* Y$ is a well-defined

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