SOME RADIUS OF CONVEXITY PROBLEMS

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1. INTRODUCTION

In a recent paper [2] the author obtained the following theorem which will be useful in certain applications in this note.

THEOREM 1. *If* $F(u, v)$ *is analytic in the* v*-plane and in the half-plane* $\Re u > 0$, *if* $P(z)$ *is regular with positive real part in* $\{|z| < 1\}$, *and if* $P(0) = 1$, *then on* $\{|z| = r < 1\}$

$$\min_{P} \min_{|z|=r} \Re F(P(z), zP'(z))$$

is attained only for a function $P = P_0$ *of the form*

$$P_0(z) = \frac{1+\alpha}{2}\left(\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\right) + \frac{1-\alpha}{2}\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right)$$

where $-1 \leq \alpha \leq 1$, $0 \leq \theta \leq 2\pi$.

The following corollary is easily verified.

COROLLARY 1. *The extremal function* P_0 *of Theorem* 1 *may be described by the equation*

$$\frac{P_0(z) - 1}{P_0(z) + 1} = \frac{bz - z^2}{1 - \bar{b}z},$$

where $b = \cos\theta + \alpha i \sin\theta$ *and* $-1 \leq \alpha \leq 1$.

It is well known [3] that if

$$f = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

maps the circle $\{|z| < 1\}$ onto a convex domain, then f is also starlike of order $1/2$; that is,

$$\Re \frac{zf'(z)}{f(z)} \geq \frac{1}{2} \qquad (|z| < 1).$$

Conversely, if f is starlike of order $1/2$ for $|z| < 1$, then it maps

$$\{|z| < (2(3)^{1/2} - 3)^{1/2} = 0.68\cdots\}$$

onto a convex domain, and the estimate is sharp. This result has been obtained just recently by T. MacGregor [1].

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