A RESULT IN THE GEOMETRY OF NUMBERS

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1. Let N denote the set of rational integers, and R the set of real numbers. Consider the function m defined on the non-negative real numbers by

$$m(c) = \max\{\min\{|\alpha - u|\,|\beta - u|;\ u \in N\}\ ;\ \alpha, \beta \in R,\ |\alpha - \beta| = 2c\}.$$

THEOREM 1. *The function* m *defined above has the following values:*

I. $$m(c) = \frac{1}{4} - c^2 \quad \text{if } 0 \le c \le \frac{1}{\sqrt{8}}.$$

II. $$m(c) = c^2 \quad \text{if } \frac{1}{\sqrt{8}} \le c \le \frac{1}{2}.$$

III. *For any positive integer* j,

$$m(c) = \begin{cases} c^2 - \dfrac{(j-1)^2}{4} & \text{if } j \le 2c \le \sqrt{j^2+1}, \\[2mm] \dfrac{(j+1)^2}{4} - c^2 & \text{if } \sqrt{j^2+1} \le 2c \le \sqrt{j^2+j+1/2}, \\[2mm] c^2 - \dfrac{j^2}{4} & \text{if } \sqrt{j^2+j+1/2} \le 2c \le j+1. \end{cases}$$

The problem of evaluating the function m was suggested to us by Professor Ivan Niven. In the ninth series of Earl Raymond Hedrick Lectures [Michigan State University, August 29 and 30, 1960, as yet unpublished], Professor Niven proved the following two lemmas:

LEMMA A. *If* β *and* α *are real numbers lying between the same pair of consecutive integers, then there exists an integer* u *such that*

$$|\beta - u|\,|\alpha - u| \le 1/4 \quad \textit{and} \quad |\beta - u| < 1.$$

LEMMA B. *If* β *and* α *are real numbers with at least one integer between them, then there exists an integer* u *such that*

$$|\beta - u|\,|\alpha - u| \le \frac{|\beta - \alpha|}{2} \quad \textit{and} \quad |\beta - u| < 1.$$

Using these two lemmas, Professor Niven constructed a very simple proof of a classical theorem of Minkowski [see 1; p. 48, Theorem IIA]. Whereas this is the major importance of these two lemmas, they also yield the result that

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