CONTINUITY PROPERTIES OF THE VISIBILITY FUNCTION

Gerald Beer

Intuitively, the visibility function for a set E in R^n measures the n-dimensional volume of the part of E visible from a variable point of E. The purpose of this paper is to establish some continuity properties of this function. The function is not as well-behaved as one might expect: we present an example of a compact plane set K with locally connected boundary of measure zero and with a discontinuity of the visibility function at an interior point. However, if K is a compact plane set with a locally connected boundary and simply connected components, the function is continuous on the interior of K. As a main result we characterize the compact sets in R^n on which the visibility function is continuous.

1. PRELIMINARIES

Definition. The *visibility function* assigns to each point x of a fixed measurable set E in a Euclidean space R^n the Lebesgue outer measure of the set

$$S(x) = \{y: rx + (1 - r)y \in E \text{ for every } r \text{ in } [0, 1]\}.$$

In [1], the reader will find a more general discussion of the visibility function and its use in describing the relative convexity of a set. In this article, we characterize the compact sets whose visibility functions are continuous, and we present sufficient conditions for the continuity of the function on compact sets and on bounded open sets in the plane.

We need three theorems established in [1].

THEOREM 1. *If* $E \subset R^n$ *is open, then the visibility function associated with* E *is lower-semicontinuous.*

THEOREM 2. *Let* $K \subset R^n$ *be compact. Then the visibility function associated with* K *is upper-semicontinuous.*

THEOREM 3. *Let* K *be a compact set in* R^n. *If* $x \in K$, *the set of all endpoints of maximal segments in* S(x) *with one endpoint at* x *forms a measurable set and has measure zero.*

We use essentially the same notation as in [1]. Ordinary Lebesgue measure in R^n is denoted by m. By $B_r(x)$ we denote the closed r-ball about a point x, and by conv E the convex hull of E. The line segment joining x to y is denoted by xy, and L(x, y) will symbolize the line determined by x and y. The symbols int E, cl E, bd E, and E^c denote as usual the interior, closure, boundary, and complement of E. Finally, the interior of a set E relative to the smallest flat containing E is denoted by intv E, and when a fixed set E is under discussion, we shall denote the visibility function for the set by the letter v.

The standard technique used to establish the lower-semicontinuity (and, hence, the continuity) of the function v at a point x of a compact set K is to show that

Received March 26, 1973.

Michigan Math. J. 20 (1973).