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THE STRONG DECIDABILITY OF CUT-LOGICS I: PARTIAL PROPOSITIONAL CALCULI

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1. *Introduction*. Recent study of partial propositional calculi in which any number of applications of the deduction rules is allowed has shown that such calculi tend to be highly undecidable; i.e., problems may be constructed concerning such calculi which are of any given degree of unsolvability (cf. [3], [4], [5]). However, in the case of calculi in which the number of applications of the deduction rules is limited, the situation is rather the reverse. It is proved below that all such calculi with finite numbers of axiom schemata and the rule modus ponens (or equivalently with finite numbers of axioms and the rules modus ponens and simultaneous substitution or substitution) are decidable and decidable in a rather strong way. The second part of this paper will concern the generalization of the decidability results to other classes of calculi (modal logics, higher order calculi, etc.)

2. *Definitions*. For the purposes of this first paper, a *partial propositional calculus* shall mean a triple $\langle M,R,N\rangle$ where M is a finite set of well-formed formulae which are theorems of the classical propositional calculus, R is either the rule modus ponens (MP) or the rule MP together with the rule simultaneous substitution (SS), or the rule MP together with the rule substitution (S) and N is a non-negative integer or infinity (∞). If N is infinity, $\langle M, \text{MP}, N\rangle$ is to be thought of as the calculus with axiom schemata corresponding to the elements of M, and sole rule MP; $\langle M, (\text{MP},\text{SS}), N\rangle$ is to be thought of as the corresponding calculus with the elements of M as axioms and MP and SS as rules, and similarly for $\langle M, (\text{MP},\text{S}), N\rangle$. The corresponding calculi when N is finite are the same calculi with the restriction that the rule MP may be applied N or fewer times only. Thus each $\langle M,R,n\rangle$ represents a subset of the calculus represented by $\langle M,R,m\rangle$ for $n < m$, and also a subset of the calculus represented by $\langle M,R,\infty\rangle$. Usually we shall identify a triple and the calculus it represents whenever this causes no confusion. The calculi with N finite will be called cut-propositional calculi.

In general, the proofs below will be carried out for the case that the connectives present are implication ($\supset$) and a constant false (f), but the modifications for other sets of connectives should be reasonably easy for

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