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ON PROVABLE RECURSIVE FUNCTIONS

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A provable recursive function is, roughly, a recursive function which is not only a total function (on the natural numbers), but can even be *proven* to be total in some formal system, e.g. first-order Peano arithmetic. In this note we discuss the problem of making this definition precise. See the references for discussion of the properties of this class of functions.

In [2] Fischer proposes that a recursive function f be called *provable* in the formal system $\mathbf{S}$ if there is an index e of f such that

$$\vdash_{\mathbf{S}} \forall x \, \exists y \, \exists z \, \mathbf{M}(e, x, y, z)$$

where $\mathbf{M}$ is a formula which binumerates (i.e. numeralwise expresses) in $\mathbf{S}$ the primitive recursive relation holding of a, b, c, d iff the a'*th* partial recursive function applied to input b gives output c in not more than d steps. The problem here is: Which such $\mathbf{M}$? If the notion does not depend on the choice of $\mathbf{M}$, the problem vanishes. But that is far from true. For simplicity assume that the formal system in question is in fact first-order Peano arithmetic $\mathbf{P}$.

Theorem: (a) *We can choose* $\mathbf{M}$ *as above such that no function is provable.*
(b) *For any total recursive* f *we can choose* $\mathbf{M}$ *as above such that* f *is provable.*

Proof: (a) Let $\mathbf{M}_0(w, x, y, z)$ be your first choice for $\mathbf{M}$. Let $\mathbf{A}(x)$ be a formula such that

$$\vdash_{\mathbf{P}} \mathbf{A}(n) \text{ for each } n, \text{ but } \nvdash_{\mathbf{P}} \forall x \, \mathbf{A}(x) \quad .$$

Let $\mathbf{M}(w, x, y, z)$ be

$$\mathbf{M}_0(w, x, y, z) \wedge \mathbf{A}(x) \quad .$$

This binumerates the same relation, but for any e

$$\nvdash_{\mathbf{P}} \forall x \, \exists y \, \exists z \, \mathbf{M}(e, x, y, z)$$

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