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SATISFIABILITY IN A LARGER DOMAIN

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The essential idea in the proof of the familiar result that a sentence which is satisfiable in some domain D is satisfiable in a larger domain D^+ $D \subset D^+$, is to define a predicate $\mathcal{P}^+$ over D^+ corresponding to a predicate $\mathcal{P}$ over D so that, for some fixed element $a \in D$,

$$\mathcal{P}^+(x_1, x_2, \ldots, x_n) = \mathcal{P}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$$

where $\bar{x}_i = x_i$, if $x_i \in D$, and $\bar{x}_i = a$, if $x_i \notin D$, $1 \leqslant i \leqslant n$.

It seems to me, however, that the application of this idea to achieve the proof is rather more difficult than the published accounts, for instance those in [1], [2] and my own [3], lead one to suppose. To complete the proof it is necessary to show that, *for any* $\mathcal{P}$, *and all sets of quantifiers* $Q_1, \ldots, Q_n$, *the sentences without free variables* $Q_n x_n Q_{n-1} x_{n-1} \ldots Q_1 x_1 \mathcal{P}^+$, $Q_n x_n Q_{n-1} x_{n-1} \ldots Q_1 x_1 \mathcal{P}$ *have the same truth value*, where each Q_i is an existential or universal quantifier and the quantifiers on $\mathcal{P}$ relate to the domain D, those on $\mathcal{P}^+$ to the domain D^+. Let us call this result (*).

We consider first the case of a single quantifier. If $(\forall x)\mathcal{P}(x)$ is true, then $\mathcal{P}(x)$ is true for any $x \in D$, and so $\mathcal{P}^+(x)$ is true for any $x \in D^+$ whence $(\forall x)\mathcal{P}^+(x)$ is true. If $(\exists x)\mathcal{P}(x)$ is true, there is an element $c \in D$ such that $\mathcal{P}(c)$ is true, and so $\mathcal{P}^+(c)$ is true, whence $(\exists x)\mathcal{P}^+(x)$ is true. If $(\forall x)\mathcal{P}(x)$ is false then $\mathcal{P}(c)$ is false for some $c \in D$, and so $\mathcal{P}^+(c)$ is false whence $(\forall x)\mathcal{P}^+(x)$ is false, and, finally, if $(\exists x)\mathcal{P}(x)$ is false then $\neg\mathcal{P}(b)$ is true for any $b \in D$, and so $\neg\mathcal{P}^+(x)$ is true for any $x \in D^+$, whence $(\exists x)\mathcal{P}^+(x)$ is false. Thus (*) holds in the case $n = 1$. Suppose then that (*) holds for any $\mathcal{P}(x_1, \ldots, x_n)$ and any set of n quantifiers; then if

$$(\forall y) Q_n x_n \ldots Q_1 x_1 \mathcal{P}(y, x_1, \ldots, x_n)$$

is true, we have $Q_n x_n \ldots Q_1 x_1 \mathcal{P}(b, x_1, \ldots, x_n)$ is true for any $b \in D$ and so by the inductive hypnothesis

$$Q_n x_n \ldots Q_1 x_1 \mathcal{P}^+(b, x_1, \ldots, x_n)$$

is true for any $b \in D$ and so for $b \in D^+$ and therefore

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