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MODELS OF $\mathrm{Th}(\langle\omega^\omega, <\rangle)$

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In this paper* we characterize the models of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$. Our main tool will be the game-theoretic characterization of elementary equivalence given by Ehrenfeucht in [2] (*cf.* also Fraissé [3]). In particular our work may be viewed as a generalization of Theorem 13 in [2] which gives a characterization of the standard, i.e., well-ordered, models of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$.

The main result, Theorem 3 of section **2**, is that a model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ consists of an ultrashort model of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ followed by at each point of an arbitrary linear order ultrashort models of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$ or of $\mathrm{Th}(\langle \ldots + \omega^n + \omega^{n-1} + \ldots + \omega + 1 + \omega^\omega, <\rangle)$, where by an ultrashort model is meant one such that for any two points x, y there is an upper bound on n such that if z is between x and y, z may be a lim_n. In Theorems 1 and 2 of section **2** we characterize ultrashort models of these two theories in terms of models of $\mathrm{Th}(\langle\omega^n, <\rangle)$. In section **1** we characterize models of $\mathrm{Th}(\langle\omega^n, <\rangle)$. In section **3** we discuss short models, namely models having no elements which are lim_n for every n. In section **4** we briefly discuss how the techniques of section **2** can be used to classify the completions of the theory of well-ordering and the element types of $\mathrm{Th}(\langle\omega^\omega, <\rangle)$.

We will assume the reader is familiar with the results and techniques in Ehrenfeucht [2]. In particular we will freely use these without further reference or mention. Several lemmas, in particular Lemmas 6, 7, 8 essentially appear in [4]. We include them for completeness and self-containment.

Our notation in general will follow that suggested in Addison, Henkin and Tarski [1]. The games $\mathbf{G}_n$ are as denoted in Ehrenfeucht [2]. We now briefly indicate our notation for linearly ordered sets:

Ordinals will be denoted as usual.
Usually if it is clear specific mention of the linear order of a linearly ordered set will be omitted.

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