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RELATIVE STRENGTH OF MALITZ QUANTIFIERS

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In this paper I will solve a problem concerning Malitz quantifiers which was posed in [1]. Before stating this problem I will introduce some notation which will be used in the proof. If X is a set then $c(X)$ is the cardinality of X and $[X]^n$ is the set of n-element subsets of X. S_n is the set of permutations of $\{1, 2, \ldots, n\}$. If $\mathfrak{A}$ is a structure $|\mathfrak{A}|$ denotes the domain of $\mathfrak{A}$. If $\mathcal{L}$ is a first-order language, $\mathcal{L}(|\mathfrak{A}|)$ is the result of adjoining to $\mathcal{L}$ one constant symbol for each element of $|\mathfrak{A}|$. No distinction will be made between elements of $|\mathfrak{A}|$ and the constant symbols denoting them. Variables will be denoted $x_1, x_2, \ldots, y_1, y_2, \ldots$.

Now let $\mathcal{L}$ be any first-order language. For each n and each infinite cardinal α a language $\mathcal{L}_\alpha^n$ is obtained from $\mathcal{L}$ by adjoining the quantifier Q_α^n with the following interpretation: $\mathfrak{A} \models Q_\alpha^n x_1 \ldots x_n \varphi(x_1, \ldots, x_n)$ if and only if there is a set $X \subset |\mathfrak{A}|$ such that $c(X) \geqslant \alpha$ and for all distinct $a_1, \ldots, a_n$ in X, $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$. Malitz and Magidor [2] and Badger [1] have established many deep and interesting results concerning these languages. In [1], page 91, Badger gave a list of unsolved problems about the languages $\mathcal{L}_\alpha^n$. There he raised the question whether $\mathcal{L}_\alpha^{n+1}$ is a proper extension of $\mathcal{L}_\alpha^n$. In this paper I answer this question affirmatively for all $n \geqslant 1$ and all $\alpha > \omega$ by exhibiting two structures $\mathfrak{A}$ and $\mathfrak{B}$ of the same similarity type such that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences in $\mathcal{L}_\alpha^n$ but do not satisfy the same sentences in $\mathcal{L}_\alpha^{n+1}$.[1]

Let n be any fixed positive integer and let α be any fixed uncountable cardinal. $\mathcal{L}$ will be a first-order language with equality whose only nonlogical symbol is an $(n+1)$-ary predicate symbol R.

Definition 1: If $\mathfrak{A}$ is an $\mathcal{L}$-structure, γ is a finite subset of $|\mathfrak{A}|$, $\sigma \in S_{n+1}$, and $t_1, \ldots, t_{n+1} \in \gamma \cup \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ then $\sigma(t_1, \ldots, t_{n+1})$ is the $(n+1)$-tuple $(t_{\sigma(1)}, \ldots, t_{\sigma(n+1)})$ and $\sigma R(t_1, \ldots, t_{n+1})$ is the $\mathcal{L}(|\mathfrak{A}|)$-formula $R(t_{\sigma(1)}, \ldots, t_{\sigma(n+1)})$.

1. For $\alpha = \omega_1$, this result was obtained independently by Andreas Baudisch under the assumption $\diamond_{\omega_1}$.

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