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The Transitivity of Implication in Tree Logic

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A *If* $\vdash S_1$ *and* $\vdash S_1 \supset S_2$ *then* $\vdash S_2$.
B *If* $S_1 \vdash S_2$ *and* $S_2 \vdash S_3$ *then* $S_1 \vdash S_3$.

These are, obviously, desirable metatheorems. The first states that the set of theorems is closed under modus ponens and one would like to appeal to A in doing a relative soundness proof, in showing, for example, that *this* system (whatever it is) is at least as strong as some other system, given by axioms and MP. (Whence, if *this* system is provably sound, so is that other.) Proposition B states that [syntactical] implication is transitive—whence the title of this paper.

These are syntactical metatheorems. (They are interderivable for tree logic without too much trouble.) What is surprising is the difficulty of getting *purely syntactical* proofs of them for tree logic. Standard procedure is to prove completeness and soundness of tree logic and then, from the corresponding semantical theorems, get quick proofs of A and B.[1]

But suppose we did the same thing for the Deduction Theorem in traditional logics, i.e., first proved completeness and soundness, without the Deduction Theorem and then got the Deduction Theorem from the corresponding semantical theorem *via* completeness and soundness. The feeling would be that we would have missed out on important insights which a purely syntactical proof of the Deduction Theorem supplies. In the same way, it seems to me, we miss out on some insights into tree logic if we do not have purely syntactical proofs of Propositions A and B. This paper aims at supplying such proofs as Metatheorems 12 and 13.

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