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Full Satisfaction Classes: A Survey

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Abstract We give a survey (with no proofs) of the theory of full satisfaction classes for models of Peano Arithmetic.

Arithmetize the language L_{PA} of Peano Arithmetic (PA). Let $M \models$ PA. By a *full satisfaction class for M* we mean a subset $S \subseteq Sent^M$ (the set of sentences in the sense of M) which satisfies the usual conditions on truth given by Tarski, i.e.

(i) If φ is of the form $S^m 0 + S^k 0 = S^r 0$ then $\varphi \in S$ iff $m + k = r$.
(ii) The same for other atomic formulas.
(iii) $(\neg\varphi) \in S$ iff $\varphi \notin S$ for each sentence φ.
(iv) $(\varphi \ \& \ \psi) \in S$ iff both φ, ψ are in S.
(v) $(\exists v_k \varphi) \in S$ iff $\varphi(S^m 0) \in S$ for some m.

Think of this as follows: S is just the notion of truth for all sentences in the sense of M, including nonstandard ones. Robinson [16] was the first to treat seriously the nonabsoluteness of the finiteness in the very definitions of the language. The notion of a full satisfaction class was defined explicitly by Krajewski [11], who proved the following nonuniqueness result.

Theorem 1 (Krajewski [11]) *There exists M, a model of* PA, *which has many different satisfaction classes. To be more specific, if S_0 is a full satisfaction class for a countable M so that (M, S_0) is recursively saturated then S_0 has $2^{\aleph_0}$ automorphic images, i.e.*

$$\{S \subseteq M : \exists g \in \mathrm{Aut}(M)\ S = g * S_0\}$$

is of power continuum.

The idea of the proof of Krajewski's result is to apply the countable version of the Chang-Makkai Theorem (cf. Schlipf [17]). In order to verify the assumption, use Tarski's Theorem on Undefinability of Truth.

After having shown that the nonstandard language determined by an $M \models$ PA does not have uniquely determined semantics, the question arises: for which models M of PA does there exist such a semantics? That is, which models admit a full satisfaction class?

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