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Polymorphism and Apartness

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Abstract Using traditional intuitionistic concepts such as apartness and subcountability, we give a relatively simple and direct construction of a natural, set-theoretic model for the second-order polymorphic lambda calculus, a model distinct from that of the modest sets.

1 Introduction The concept of apartness is an intuitionistic "positivization" of the classical notion of inequality between real numbers. In classical mathematics, every set has an apartness defined over it: apartness and inequality coincide. Intuitionistically, things can be *very* different—it is consistent with the full intutionistic set theory IZF to assume that every apartness space is *subcountable*, i.e., a quotient of a set of natural numbers. What follows almost immediately from this is a relatively simple and direct set-theoretic construction of a natural model for the second-order polymorphic lambda calculus $\mathbf{P}\lambda$.

Working within the Kleene realizability universe $\mathcal{V}(Kl)$ for set theory, we construct a small category $\mathcal{C}$ of sets which allow apartness and which, thanks to the presence of local axioms of choice, constitute a natural model of $\mathbf{P}\lambda$. This affords us another clear indication of the mathematical advantages of intuitionistic over classical metamathematics: using classical metamathematics, Reynolds (in [27]) has shown that, on pain of violating Cantor's uncountability theorems, there can be no natural set-theoretic models of $\mathbf{P}\lambda$.

Our construction is one of a number of intuitionistic models for $\mathbf{P}\lambda$ (cf. Pitts [24], Longo and Moggi [16]). The most popular of these is constructed over the category of realizability-valued modest sets $\mathcal{M}$. The model $\mathcal{C}$ of the present paper is distinct from that of modest sets: we prove that $\mathcal{C}$ is a proper subcategory of $\mathcal{M}$ in that the set of objects of the former is a proper subset of that of the latter. Second—and more importantly—our model construction does not leave one with a faulty impression that has been fostered, we think, by the details of the mathematics of $\mathcal{M}$: that the existence of models of the polymorphic lambda cal-

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