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Some Compactness Results for Modal Logic

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Abstract A modal logic L is said to be compact if every L-consistent set of formulas has a model on a frame for L. Some large classes of compact (noncompact) logics are identified, and it is shown that there are uncountably many compact (noncompact) logics.

A modal logic L is *compact* if every L-consistent set of formulas has a model on a frame for L, *classically compact* if a set of formulas fails to have a model on a frame for L only if some finite subset fails to have a model on a frame for L, *canonical* if determined by its canonical frame, and *complete* if determined by some class of frames. These four properties are related in an obvious way:

$$\begin{array}{c} \text{CANONICALCY} \Rightarrow \text{COMPACTNESS} \Rightarrow \text{COMPLETENESS} \\ \Downarrow \\ \text{CLASSICAL COMPACTNESS} \end{array}$$

Given the amount of attention that has been lavished upon canonicalcy and completeness, it is therefore mildly surprising that compactness has enjoyed relatively little press. The first explicit mention of it would seem to be found in Corcoran and Weaver [1]. However, they were working with a different concept of a model than the now standard one used here, and as a result the nice connection between canonicalcy and compactness is lost. (See [2], in which they show that the canonical logics T and B are, on their account, noncompact.) Fine raises the issue of compactness at the end of [5], but his important work on this notion did not appear in print until more than a decade later with the publication of [6]. Even then, although he was undoubtedly aware that his argument could be generalized, only one example of a familiar, noncompact logic is actually mentioned. Hughes and Cresswell [10] go a bit further, giving several examples. Unfortunately, their proofs (and suggested proofs) contain a seductive error, and

*An earlier version of this paper was presented to the Annual Meeting of the Association for Symbolic Logic held in Chicago, Illinois, April 26–27, 1985. With the exception of Theorem 4, these results were obtained before I saw [6], [10], and [11].

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