Notre Dame Journal of Formal Logic
Volume 30, Number 1, Winter 1989

The Elementary Theory of the Natural Lattice Is Finitely Axiomatizable

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Introduction It is well known that the set of positive integers with the divisibility relation is a lattice, indeed the prototype of lattices. Here we call it the *natural lattice*. What are the differences between this lattice and other lattices? What are the particular properties (in the language of lattices, defined below) of this lattice? How should it be characterized?

Some linear orders have been studied (the natural order of the positive integers by Dedekind in [5], the orders of rationals and of reals by Cantor in [2], see also [9]). But no characterizations exist for particular lattices.

A mathematical characterization exists for $(\mathbb{N}^*,/)$. It is a partial order with a least member, 1, a denumerable set of atoms (the prime numbers), each member x has a *p-successor* for each atom p (the product $p \cdot x$), and the following *multi-induction principle*: a subset A of $\mathbb{N}^*$ which contains 1, and is such that if x belongs to A then $p \cdot x$ belongs to A for all atoms p, is $\mathbb{N}^*$. But this characterization is not in the hierarchy of logical languages (first-order, second-order, . . .). (In particular, this characterization is not expressible in a second-order language because of the denumerability of the set of atoms).

The logical language of the theory of lattices is naturally the first-order language of partial order, with only a binary predicate, denoted by $\leq$. Our aim is to characterize (i.e., to axiomatize) the first-order theory DIV of the structure $(\mathbb{N}^*,/)$. DIV is consistent, complete, but not $\aleph_0$-categorical (the standard model is not the only countable model). This theory is decidable (stated by Skolem in [13], but proved first by Mostowski in [12]), thus recursively axiomatizable. But the computational complexity of the axiomatization given by this method is very awkward. We show that this theory is finitely axiomatizable, giving an explicit finite axiomatization. This fact seems prominent because relatively few theories of structures are finitely axiomatizable. The theory of addition and the theory of multiplication are not (see [3]).

Received May 19, 1986; revised January 19, 1987