ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 3, 2007

ON SOME TOPOLOGICAL PROPERTIES OF VECTOR-VALUED FUNCTION SPACES

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ABSTRACT. Let E be an ideal of L^0 over a σ-finite measure space (Ω, Σ, μ) with a Hausdorff locally convex-solid topology ξ, and let $(X, \|\cdot\|_X)$ be a real Banach space. Let $E(X)$ be a subspace of the space $L^0(X)$ of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$ and consisting of all those $f \in L^0(X)$ for which the scalar function $\|f(\cdot)\|_X$ belongs to E. In this paper we show that a number of topological properties of the spaces X and (E, ξ) can be lifted to the space $(E(X), \bar{\xi})$, where $\bar{\xi}$ stands for the topology on $E(X)$ associated with ξ. We characterize some important topological properties of the space $(E(X), \bar{\xi})$ (weak compactness of order intervals, almost reflexivity, weak sequential completeness, semi-reflexivity, relative weak compactness of solid hulls) in terms of the corresponding properties of X and (E, ξ).

1. Introduction and preliminaries. Let E be an ideal of L^0 (over a σ-finite measure space) with a Hausdorff locally convex-solid topology ξ, and let X be a real Banach space. The aim of this paper is to extend some important topological properties of the space (E, ξ) to the vector-valued function space $(E(X), \bar{\xi})$, where $\bar{\xi}$ stands for the topology on $E(X)$ associated with ξ. We characterize the following topological properties of the space $(E(X), \bar{\xi})$: weak compactness of order intervals: Section 2, almost reflexivity; Section 3, weak sequential completeness; Section 4, semi-reflexivity; Section 5, relative weak compactness of solid hull; Section 6, in terms of the corresponding properties of X and (E, ξ).

In the particular case of E being a Banach function space, over a finite measure space, the problem of characterizing the topological properties of the Köthe-Bochner space $E(X)$ in terms of the properties of both Banach spaces E and X has been considered by Pisier [**28**], Bombal

2000 AMS *Mathematics Subject Classification.* Primary 46E40, 46E30, 46A20, 46A25.
Key words and phrases. Vector-valued function spaces, Köthe-Bochner spaces, locally solid topologies, Lebesgue topologies, order intervals, conditional weak compactness, weak compactness, weak sequential completeness, almost reflexivity, semi-reflexivity, relative weak compactness of solid hulls.
Received by the editors on Feb. 25, 2004, and in revised form on Jan. 26, 2005.