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INTEGRAL CLOSURES, LOCAL COHOMOLOGY AND IDEAL TOPOLOGIES

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ABSTRACT. Let $(R, \mathfrak{m})$ be a formally equidimensional local ring of dimension d. Suppose that Φ is a system of nonzero ideals of R such that, for all minimal prime ideals $\mathfrak{p}$ of R, $\mathfrak{a}+\mathfrak{p}$ is $\mathfrak{m}$-primary for every $\mathfrak{a} \in \Phi$. In this paper, the main result asserts that for any ideal $\mathfrak{b}$ of R, the integral closure $\mathfrak{b}^{*(H^d_\Phi(R))}$ of $\mathfrak{b}$ with respect to the Artinian R-module $H^d_\Phi(R)$ is equal to $\mathfrak{b}_a$, the classical Northcott-Rees integral closure of $\mathfrak{b}$. This generalizes the main result of [**13**] concerning the question raised by D. Rees.

1. Introduction. Let R denote a commutative Noetherian ring (with identity) of dimension d, and let A be an Artinian R-module. We say that the ideal $\mathfrak{a}$ of R is a *reduction* of the ideal $\mathfrak{b}$ of R with respect to A if $\mathfrak{a} \subseteq \mathfrak{b}$ and there exists an integer $s \geq 1$ such that $(0 :_A \mathfrak{a}\mathfrak{b}^s) = (0 :_A \mathfrak{b}^{s+1})$. An element x of R is said to be *integrally dependent on* $\mathfrak{a}$ *with respect to* A if $\mathfrak{a}$ is a reduction of $\mathfrak{a} + Rx$ with respect to A, see [**12**]. Moreover, the set $\mathfrak{a}^{*(A)} := \{x \in R \mid x$ is integrally dependent on $\mathfrak{a}$ with respect to $A\}$ is an ideal of R, *called the integral closure of* $\mathfrak{a}$ with respect to A.

In [**13**] the dual concepts of reduction and integral closure of the ideal $\mathfrak{b}$ with respect to a Noetherian R-module N were introduced; we shall use $\mathfrak{b}_a^{(N)}$ to denote the integral closure of $\mathfrak{b}$ with respect to N. If $N = R$, then $\mathfrak{b}_a^{(N)}$ reduces to that the usual Northcott-Rees integral closure $\mathfrak{b}_a$ of $\mathfrak{b}$.

The purpose of the present paper is to show that, for any system of ideals Φ of a formally equidimensional local ring $(R, \mathfrak{m})$ of dimension d, if $\operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$ for all minimal primes $\mathfrak{p}$ of R and for every $\mathfrak{a} \in \Phi$, then $\mathfrak{b}^{*(H^d_\Phi(R))}$, the integral closure of $\mathfrak{b}$ with respect to $H^d_\Phi(R)$, is equal

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