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A LITTLE WEDDERBURN PRINCIPAL THEOREM

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ABSTRACT. We point out that, even if $A/\mathrm{rad}\, A$ is inseparable, the finite-dimensional algebra (or, more generally, left Artinian ring) A admits a canonical ideal direct sum decomposition $A = A^0 \oplus A^\#$ such that $\mathrm{rad}\, A^0 = (0)$, $\mathrm{rad}\, A^\# = \mathrm{rad}\, A$, $\mathrm{rad}\, A^\#$ is an essential ideal of $A^\#$, and $A^\#$ is unital if and only if A is unital. This is a consequence of a general and maximally elementary splitting of an arbitrary ring relative to a suitable nil ideal. This process of *essentializing* a nil ideal is useful in the study of categories of ideal and radical embeddings.

1. Background. We begin with an associative algebra A, not necessarily unital, which is finite-dimensional over the field k. Let N denote the nilpotent radical $\mathrm{rad}\, A$ of A. Suppose we wish to explore the interactions of N with other parts of A (the third main problem in the study of algebras). How might the standard theory guide us?

The celebrated Wedderburn principal theorem [**1**, Theorem 3.23; **2**, Theorem 72.19; **5**, Theorem 11.6] assures us that if the semi-simple k-algebra A/N is separable, in particular, if the scalar field k is perfect, then A contains at least one separable subalgebra (*Wedderburn factor*) S such that

$$A = S \dotplus N \qquad (k\text{-direct sum}). \tag{1.1}$$

Next, by Wedderburn's comparatively elementary results on semi-simple algebras, we have a canonical decomposition of S into two-sided ideals

$$S = S_0 \oplus S_1 \qquad (S\text{-direct sum})$$

provided we define $S_0 = \mathrm{ann}_S(N) =$ the two-sided annihilator of the S-bimodule N. Thus, S_0 is either zero or semi-simple and $S_1 = \mathrm{ann}_S(S_0)$ is the unique multiplicatively orthogonal complement of S_0 in S.

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