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SHARP INEQUALITIES FOR THE HURWITZ ZETA FUNCTION

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ABSTRACT. We prove the following double-inequality for the Hurwitz zeta function $\zeta(p,a)=\sum_{\nu=0}^{\infty}(\nu+a)^{-p}$.

Let m and n be integers with $m>n\geq 0$ and let a be a positive real number. Then we have for all real numbers $p>1$:

$$\frac{m+1+a}{n+1+a}<\left(\frac{\zeta(p,a)-\sum_{\nu=0}^{n}(\nu+a)^{-p}}{\zeta(p,a)-\sum_{\nu=0}^{m}(\nu+a)^{-p}}\right)^{1/(p-1)}$$
$$<\exp\biggl(\sum_{\nu=n+1}^{m}\frac{1}{\nu+a}\biggr).$$

Both bounds are best possible.

Our theorem extends and refines a result of Bennett [**2**].

1. Introduction. In order to prove a sharp lower bound for the Cesàro matrix, Bennett [**2**] applied the following inequality for the "tail" of the series representation of the classical Riemann zeta function:

$$f_p(n)<f_p(n+1),\quad n=1,2,\dots,$$

where

$$f_p(n)=n^{p-1}\sum_{\nu=n+1}^{\infty}\nu^{-p},\quad p>1.$$

The monotonicity of f_p provides an interesting upper bound for the ratio $\left(\sum_{\nu=n+1}^{\infty}\nu^{-p}/\sum_{\nu=m+1}^{\infty}\nu^{-p}\right)^{1/(p-1)}$, which does not depend on p:

$$\left(\frac{\zeta(p)-\sum_{\nu=1}^{n}\nu^{-p}}{\zeta(p)-\sum_{\nu=1}^{m}\nu^{-p}}\right)^{1/(p-1)}<\frac{m}{n},\quad p>1;\ m>n\geq 1. \tag{1.1}$$

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