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A QUARTIC SURFACE OF INTEGER HEXAHEDRA

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ABSTRACT. We prove that there are infinitely many six-sided polyhedra in $\mathbf{R}^3$, each with four congruent trapezoidal faces and two congruent rectangular faces, so that the faces have integer sides and diagonals, and also the solid has integer length diagonals. The solutions are obtained from the integer points on a particular quartic surface.

A long standing unsolved problem asks whether or not there can be a parallelepiped in $\mathbf{R}^3$ whose sides and diagonals have integer length. If one weakens the requirement and just asks for a six-sided polyhedron with quadrilateral faces, then one can find examples with integer length sides and diagonals. Peterson and Jordan [**1**] described a method for making these 'perfect' hexahedra. We review their method.

Take two congruent rectangles positioned as if they formed the top and bottom parallel faces of a rectangular box. Rotate the top rectangle by 90 degrees around the axis joining the centers of these two rectangles. Now connect the sides of the two rectangles with four congruent trapezoids. (The shape can be viewed as a piecewise linear version of the placement of two cupped hands together, at 90 degrees in clapping position.) If the sides of the rectangle have lengths a, b, then the diagonal has length c, where $a^2 + b^2 = c^2$. The parallel sides of the trapezoids are then also a, b. If the slant side of the trapezoid is, say, e and its diagonal is d, then it follows from Ptolemy's theorem that $d^2 = e^2 + ab$. Consider the trapezoid with base on the top rectangle of side a and other base on the bottom rectangle opposite that edge of side b having the slant sides of length d. Its diagonal is of length f, and also it is the interior diagonal of the hexahedron; thus, $f^2 = d^2 + ab$. We shall refer to such polyhedra, for which these six parameters are integral, as perfect hexahedra.

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