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VARIATIONS ON TWISTS OF ELLIPTIC CURVES

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ABSTRACT. In this note we show that for any triple E_1, E_2, E_3 of elliptic curves, where the j-invariants of the curves E_2, E_3 are equal to 0, there exist rational functions $D_{2,3,3}$, $D_{2,6,6} \in \mathcal{Q}(u,v,w)$ with the following properties:

- the quadratic twist of the curve E_1 and the cubic twists of the curves E_2, E_3 by the $D_{2,3,3}$ have positive rank over $\mathcal{Q}(u,v,w)$,
- the quadratic twist of the curve E_1 and the sextic twists of the curves E_2, E_3 by the $D_{2,6,6}$ have positive rank over $\mathcal{Q}(u,v,w)$.

Moreover, we also prove that if the j-invariant of E_1 is equal to 0, then there exists a rational function $D_{3,3,6} \in \mathcal{Q}(u,v,w)$ with the property that the cubic twists of the curves E_1, E_2 and the sextic twists of the curve E_3 by $D_{3,3,6}$ have positive rank over $\mathcal{Q}(u,v,w)$.

1. Introduction. Let E_1, E_2 be elliptic curves with the property that their j-invariants are not equal to 0 or 1728 simultaneously. Kuwata and Wang in the paper [**4**] proved the existence of a polynomial D such that the quadratic twist $E_{i,\,D}$ of the curve E_i by D has positive rank for $i = 1, 2$. Their method cannot be used in the case when $j(E_1) = j(E_2) = j$, for $j = 0, 1728$. Unfortunately, we are unable to show that in these cases it is also possible to construct quadratic twists of pairs of elliptic curves with positive rank. It is known that each elliptic curve has a quadratic twist. However, it is well known, that elliptic curves with the j-invariant equal to 0 or, in other words, curves of the form $E : y^2 = x^3 + p$, also have higher twists. The cubic twist of the curve E by D has the equation $y^2 = x^3 + pD^2$. However, for our purposes, it will be more convenient to work with the isomorphic model of the cubic twist given by the equation $y^2 = Dx^3 + p$. The sextic twist of the curve E by D is given by the equation $y^2 = x^3 + pD$.

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