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DISTINGUISHED ORBITS OF REDUCTIVE GROUPS

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ABSTRACT. We prove a generalization of a theorem of Borel-Harish-Chandra on closed orbits of linear actions of reductive groups. Consider a real reductive algebraic group G acting linearly and rationally on a real vector space V. The group G can be viewed as the real points of a complex reductive group $G^{\mathbf{C}}$ which acts on $V^{\mathbf{C}} := V \otimes \mathbf{C}$. In [**2**] it was shown that $G^{\mathbf{C}} \cdot v \cap V$ is a finite union of G-orbits; moreover, $G^{\mathbf{C}} \cdot v$ is closed if and only if $G \cdot v$ is closed, see [**20**]. We show that the same result holds not just for closed orbits but for the so-called distinguished orbits. An orbit is called *distinguished* if it contains a critical point of the norm squared of the moment map on a projective space. Our main result compares the complex and real settings to show that $G \cdot v$ is distinguished if and only if $G^{\mathbf{C}} \cdot v$ is distinguished.

In addition, we show that, if an orbit is distinguished, then under the negative gradient flow of the norm squared of the moment map, the entire G-orbit collapses to a single K-orbit. This result holds in both the complex and real settings.

We finish with applications to the study of left-invariant geometry of Lie groups; of particular interest are left-invariant Einstein and Ricci soliton metrics on solvable and nilpotent Lie groups. Using the above theorems, we obtain a procedure for recovering Ricci soliton metrics on nilpotent Lie groups.

1. Introduction. An analytic approach to finding closed orbits in the complex setting was developed by Kempf and Ness [**19**] and extended to the real setting by Richardson and Slodowy [**20**]. From their perspective, the closed orbits are those that contain zeros of the so-called moment map. However, one can consider more generally critical points of this moment map on projective space. Work on the moment map in the complex setting has been done by Ness [**18**] and Kirwan [**10**]. Following those works, the real moment map was explored by Marian [**15**] and Eberlein and Jablonski [**3**].

Consider a real linear reductive group G acting linearly and rationally on a real vector space V. There is a complex linear reductive group $G^{\mathbf{C}}$ such that G is a finite index subgroup of the real points of $G^{\mathbf{C}}$;

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