ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 4, 2012

GEOMETRY OF CANONICAL SELF-SIMILAR TILINGS

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ABSTRACT. We give several different geometric characterizations of the situation in which the parallel set F_ε of a self-similar set F can be described by the inner ε-parallel set $T_{-\varepsilon}$ of the associated canonical tiling $\mathcal{T}$, in the sense of [**15**]. For example, $F_\varepsilon = T_{-\varepsilon} \cup C_\varepsilon$ if and only if the boundary of the convex hull C of F is a subset of F, or if the boundary of E, the unbounded portion of the complement of F, is the boundary of a convex set. In the characterized situation, the tiling allows one to obtain a tube formula for F, i.e., an expression for the volume of F_ε as a function of ε. On the way, we clarify some geometric properties of canonical tilings.

Motivated by the search for tube formulas, we give a generalization of the tiling construction which applies to all self-affine sets F having empty interior and satisfying the open set condition. We also characterize the relation between the parallel sets of F and these tilings.

1. Introduction. As the basic object of our study is a self-affine system and its attractor, the associated self-affine set, we begin by defining these terms.

Definition 1.1. For $j = 1, \dots, N$, let $\Phi_j : \mathbf{R}^d \to \mathbf{R}^d$ be an affine contraction whose eigenvalues λ all satisfy $0 < |\lambda| < 1$. Then $\{\Phi_1, \dots, \Phi_N\}$ is a *self-affine iterated function system.*

Definition 1.2. A *self-similar system* is a self-affine system for which each mapping is a similitude, i.e.,

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2010 AMS *Mathematics subject classification.* Primary 28A80, 28A75, 52A20, 52C22, Secondary 52A38, 53C65, 51M25, 49Q15, 60K05, 54F45.
Keywords and phrases. Iterated function system, parallel set, fractal, complex dimensions, zeta function, tube formula, Steiner formula, renewal theorem, convex ring, inradius, Euler characteristic, Euler number, self-affine, self-similar, tiling, curvature measure, generating function, fractal string.
The work of the first author was partially supported by the University of Iowa Department of Mathematics NSF VIGRE grant DMS-0602242. The work of the second author was partially supported by Cornell University and a grant from the German Academic Exchange Service (DAAD).
Received by the editors on February 3, 2009, and in revised form on May 18, 2009.
DOI:10.1216/RMJ-2012-42-4-1327