ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 42, Number 3, 2012

HARMONIC MAPS AND KALUZA-KLEIN METRICS ON SPHERES

M. BENYOUNES, E. LOUBEAU AND L. TODJIHOUNDE

ABSTRACT. This article studies the harmonicity of vector fields on Riemannian manifolds, viewed as maps in the tangent bundle equipped with a family of Riemannian metrics. Geometric and topological rigidity conditions are obtained, especially for surfaces and vector fields of constant norm, and existence is proved on two-tori. Classifications are given for conformal, quadratic and Killing vector fields on spheres. Finally, the class of metric considered on the tangent bundle is enlarged, permitting new vector fields to become harmonic.

1. Introduction. Though very interesting in many settings, the theory of harmonic maps fails to produce any worthwhile result when applied to vector fields, seen as maps from a Riemannian manifold (M, g) into its tangent bundle TM equipped with its simplest metric, the Sasaki metric. This situation has led researchers to consider constrained problems on the same functional, e.g., harmonic sections and harmonic unit sections. However, recently, new classes of metrics on TM have been shown to allow a richer existence theory and, with respect to adequate metrics, standard vector fields can produce new harmonic maps, for example, a two-parameter family including the Sasaki metric [**4**], g-natural metrics [**1**] or an ad-hoc Riemannian metric based on a deformation of the horizontal distribution [**17**].

The main difficulty here is to strike a balance between the harmonicity of vector fields and the geometric relevance of the metric on TM. In this paper, given a Riemannian manifold (M, g), we consider on TM Riemannian metrics in the intersection of the largest known class of metrics on tangent bundles, i.e., g-natural metrics and Kaluza-Klein metrics, as commonly defined on principal bundles (cf. [**19**]).

Recall that, at a point $(p, e) \in TM$, the tangent space $T_{(p,e)}TM$ splits into its horizontal and vertical spaces [**8**]:

2010 AMS *Mathematics subject classification.* Primary 58E20.
Keywords and phrases. Harmonic sections, harmonic maps, tangent bundle.
Received by the editors on March 31, 2009, and in revised form on November 3, 2009.

DOI:10.1216/RMJ-2012-42-3-791