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EXPLICIT ELLIPTIC $K3$ SURFACES WITH RANK 15

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ABSTRACT. This note presents a relatively straightforward proof of the fact that, under certain congruence conditions on $a, b, c \in \mathbf{Q}$, the group of rational points over $\overline{\mathbf{Q}}(t)$ on the elliptic curve given by

$$y^2 = x^3 + t^3(t^2 + at + b)^2(t + c)x + t^5(t^2 + at + b)^3$$

is trivial. This is used to show that a related elliptic curve yields a free abelian group of rank 15 as its group of $\overline{\mathbf{Q}}(t)$-rational points.

1. Introduction. The theory of elliptic curves defined over the function field $k(C)$ of a curve C/k is quite rich. To a large extent, this is due to the observation that any such elliptic curve $E/k(C)$ corresponds to a minimal k-morphism $\pi : \mathcal{E} \to C$ in which $\mathcal{E}$ is a smooth surface over k, and the generic fiber of π is isomorphic to E. Rational points on E correspond to sections of π, and the geometry of $\mathcal{E}$ gives a better understanding of the Mordell-Weil group $E(k(C))$. An exposition of this theory is given in [**7**].

For example, when $\mathcal{E}$ is a rational surface and k is separably closed, one has the Shioda-Tate formula

$$\operatorname{rank} E(k(C)) = 8 - \sum_{\nu} (m_\nu - 1),$$

where m_ν is the number of irreducible components of the fiber $\pi^{-1}(\nu)$ over $\nu \in C$. Using the formula, it is easy to construct explicit examples with a given rank r satisfying $0 \le r \le 8$. This is done in the table below, at least over $\mathbf{C}(t)$.

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