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THE ULTRAFILTER THEOREM IN REAL ALGEBRAIC GEOMETRY

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Introduction. Let R be a real closed field and let $V \subset R^n$ be a real algebraic set, $A(V) = R[x_1, \ldots, x_n]/I(V)$ the affine coordinate ring of V. The ultrafilter theorem says there is a natural bijective correspondence between ultrafilters of semi-algebraic subsets of V and points of the real spectrum of $A(V)$, [**1**, **2**].

The real spectrum, $\mathrm{Spec}_R(A)$, of a commutative ring A can be identified with, or defined as, the collection of prime cones in A: that is, subsets, α, of A which satisfy (i) $\alpha + \alpha \subset \alpha$, (ii) $\alpha \cdot \alpha \subset \alpha$, (iii) $\Sigma A^2 \subset \alpha$, where ΣA^2 denotes the sums of squares in A, (iv) $-1 \notin \alpha$, (v) $\alpha \cup -\alpha = A$, and (vi) $\alpha \cap -\alpha = p(\alpha)$ is a prime ideal of A. Given such an α, the residue ring $A/p(\alpha)$ is totally ordered, with non-negative elements being the image of α. Conversely, given a total ordering on A/p, $p \subset A$ a prime ideal, the inverse image, α, of its non-negative elements satisfies (i)-(vi). Of course, total orderings of rings must be compatible with the arithmetic operations in the usual way. Note $V \subset \mathrm{Spec}_R[A(V)]$, since a point of V can be identified with a maximal ideal of $A(V)$ with residue ring R.

Given an ultrafilter of semi-algebraic subsets of V, define $\alpha \subset A(V)$ by $f \in \alpha$ if $f(x) \geq 0$, all $x \in C$, for some semi-algebraic $C \subseteq V$ which belongs to the ultrafilter. Then one can show $\alpha \in \mathrm{Spec}_R[A(V)]$, and this is the correspondence of the ultrafilter theorem.

By a constructible subset of $\mathrm{Spec}_R(A)$, we mean any member of the smallest family of subsets closed under finite intersections, finite unions, and complements, and containing the sets $W(f) = \{\alpha \in \mathrm{Spec}_R(A) | f \in \alpha\}$. Note that $f \in \alpha$ just says that the image, $f(\alpha)$, of f in $A/p(\alpha)$ is non-negative. If $x \in V \subset \mathrm{Spec}_R[A(V)]$, then $x \in W(f)$ says $f(x) \geq 0 \in R$.

We offer the following proof of the ultrafilter theorem, which has certainly been noticed by others, for example, L. van den Dries, M.

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