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# PIECEWISE-RATIONAL RETRACTIONS ONTO CLOSED, CONVEX, SEMI-ALGEBRAIC SETS WITH INTERIOR-SYNOPSIS

CHARLES N. DELZELL

Dedicated to the memory of Gus Efroymson

Let $(K, <)$ be an ordered field, contained in a real closed order-extension field $R$. Let $X = (X_1, \ldots, X_n)$ be indeterminates and let $x = (x_1, \ldots, x_n) \in R^n$. A set $A \subseteq R^n$ is called semi-algebraic (abbreviated s.a.: more precisely, $K$-$R$-s.a.) if it is a finite union of finite intersections of sets (and of complements of sets) of the form $\{x \in R^n | f(x) > 0\}$, $f \in K[X]$. Similarly for subsets of $R^m$, $m \neq n$. If $A \subseteq R^n$ and $B \subseteq R^m$ are s.a., and if $L$ is a subfield of $R$, then a function $f$: $A \to B$ will be called an $L$-function if $f$ takes points of $A$ with coordinates in $L$ ("$L$-rational points") to points of $B$ with coordinates in $L$; i.e., if $f(A \cap L^n) \subseteq L^m$.

DEFINITION. We shall call a function $f = (f_1, \ldots, f_m)$, from a ($K$-$R$-) s.a. set $A$ in $R^n$ to a ($K$-$R$-) s.a. set in $R^m$, ($K$-$R$-) piecewise-rational, abbreviated ($K$-$R$-) p.r., if we can decompose $A$ into a finite number of ($K$-$R$-) s.a. sets $W_i$, $A = \bigcup_i W_i$, such that for each $i$ and for $1 \leqq j \leqq m$, there is a rational function in $K(X)$ which agrees with $f_j$ on $W_i$.

The absolute value function $x \mapsto |x|$ is a good example of a (continuous) $\mathbf{Q}$-$R$-p.r. function from $R^1$ to $R^1$. Of course, all rational functions are also p.r. Clearly, $K$-$R$-p.r. functions are $L$-functions, uniformly for all fields $L$ between $K$ and $R$ (i.e., for $K \subseteq L \subseteq R$).

DEFINITION. A $K$-$R$-s.a. set $S$ is a $K$-$R$-p.r.-neighborhood-retract if there exists an open $K$-$R$-s.a. neighborhood $U \supseteq S$ and a retraction $r$: $U \to S$ which is $K$-$R$-p.r.

We may as well require $U$ to be regular (i.e., equal to the interior of its closure), since we can shrink it if necessary until it is regular, by triangulating $U$ and $S$ and subdividing.

Recall that an ordered field $K$ is called Archimedean (over $\mathbf{Q}$) if for all $d \in K$ there exists $e \in \mathbf{Q}$ such that $d < e$ (e.g., $\mathbf{Q}$ and $\mathbf{R}$ are Archimedean). We can now state the main theorem.

RETRACTION THEOREM. *Let $K$ be Archimedean. Let $W \subseteq R^n$ be a closed,*