ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

AN INTRODUCTION TO REAL ALGEBRA

T. Y. LAM

Dedicated to the memory of Gus Efroymson

Introduction. Real Algebra, roughly speaking, is the study of "real" objects such as real rings, real places and real varieties. Becuase of the recent interest in developing algebraic geometry over the real numbers (or, more generally, over a real closed field), the algebraic study of these real objects has attracted considerable attention. In many ways, the role played by real algebra in the development of real algebraic geometry is analogous to the role played by commutative algebra in the development of classical algebraic geometry. Therefore, real algebra, like commutative algebra, is a subject with a great potential for applications to geometric problems.

In the category of fields, the real objects (namely, the formally real fields) have been studied already a long time ago by Artin and Schreier, who recognized that formally real fields are precisely the fields which can be ordered. The idea of exploiting the orderings in a real field, for instance, was central in Artin's solution to Hilbert's 17th Problem. By the 70's, there was already a sizable literature on formally real (or ordered) fields. However, real algebra in the category of rings underwent a much slower development. There are several possible notions of reality for rings and people weren't sure which one to adopt. Similarly, it was not at all clear how one should define for rings the notion of orderings. Fortunately, with the impetus given by real algebraic geometry, these problems have recently been successfully resolved. There is now a consensus (or almost a consensus) about what an ordering on a ring should be, and, with this notion of orderings, there is a remarkably complete analogue of the Artin-Schreier theory valid for rings. Further, Coste and Coste-Roy have introduced the notion of the real spectrum of a ring. On the one hand, this is the correct generalization of the space of orderings of a field, and on the other hand, this offers a "real" analogue of the Zariski prime spectrum of a ring. With the discovery of this notion, one seems to be now fully ready for a

This report was done with support of an NSF grant at the Center for Pure and Applied Mathematics at U.C. Berkeley.