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LINEAR MONOTONE METHOD FOR NONLINEAR BOUNDARY VALUE PROBLEMS IN BANACH SPACES

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. One of the most useful techniques in proving the existence of multiple solutions of nonlinear boundary value problems (BVP for short) is the monotone iterative method, which yields monotone sequences that converge to extremal solutions of the problem. Recently, because of applications, this technique has attracted much attention, see **[1, 2, 4-7, 11, 12, 14, 15]**. To explain this method, let us consider the scalar BVP

$$
\begin{aligned}
&u'' = f(t, u, u'), \ 0 < t < 1,\\
&B^i u = \alpha_i u(i) + (-1)^{i+1}\beta_i u'(i) = b_i, \ i = 0, 1,
\end{aligned}
\tag{1.1}
$$

where $\alpha_i, \beta_i \geqq 0$, $\alpha_i^2 + \beta_i^2 \neq 0$ and $f \in C[I \times \mathbf{R} \times \mathbf{R}, \mathbf{R}]$, I being the interval $[0,1]$. Suppose that $v_0, w_0 \in C^2[I, \mathbf{R}]$ with $v_0(t) \leqq w_0(t)$ on I and

$$
\begin{aligned}
&v_0'' \geqq f(t, v_0, v_0'), \ B^i v_0 \leqq b_i,\\
&w_0'' \leqq f(t, w_0, w_0'), \ B^i w_0 \geqq b_i.
\end{aligned}
\tag{1.2}
$$

Then v_0, w_0 are called lower and upper solutions of (1.1). Suppose also that f_u, $f_{u'}$ exist and f satisfies a Nagumo condition. In order to obtain monotone iterations, one considers the auxiliary BVP

$$
u'' = F(t, u, u'), \ B^i u = b_i \tag{1.3}
$$

where $F(t, u, u') = f(t, \eta(t), u') + M_1(c)(u - \eta(t))$, $v_0(t) \leqq \eta(t) \leqq w_0(t)$, $|f_u(t, u, u')| \leqq M(c)$ for $t \in I$, $v_0(t) \leqq u \leqq w_0(t)$ and $|u'| \leqq c$ for some suitable $c > 0$ which is related to the Nagumo constant. To proceed further with the monotone method it becomes necessary to show that there exists a unique solution for the BVP (1.3). For this purpose, one proves that (i) v_0, w_0 are also lower and upper solutions of (1.3) and (ii) F satisfies a Nagumo condition. Then it follows from known results **[3, 8, 9, 10, 12,**

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