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DIFFUSION IN FISHER'S POPULATION MODEL

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ABSTRACT. We consider Fisher's selection model for a single locus with n alleles. The population is distributed in a spatial domain, migration is represented by a diffusion term. For the resulting parabolic system with zero flux boundary conditions we show that if the selection model has a stable polymorphism, then every solution of the parabolic system converges to a spatially homogeneous stationary solution. If initially all genes are present in the population, then this solution is the polymorphism. The results carry over to some types of convolution equations.

Diffusion in Fisher's population model. We consider the well-known Fisher-Wright-Haldane model of population genetics for n alleles a_1, a_2, $\ldots$, a_n, where p_j is the frequency of the j-th allele, and $f_{jk} = f_{kj} > 0$ is the fitness (viability, malthusian parameter) of the genotype $a_j a_k$. We assume continuous time, i.e., overlapping generations. The differential equations read

$$(1) \qquad \dot{p}_j = \sum_{k=1}^{n} f_{jk} p_k p_j - \sum_{r,s=1}^{n} f_{rs} p_r p_s p_j, \quad j = 1, 2, \ldots, n.$$

For the derivation of the model and for results on the ordinary differential equation we refer to [1], [2], [4], or [3]. The last reference contains an account of the earlier literature and more detailed results.

We prefer to use a condensed notation. Let $F = (f_{jk})$ be the matrix of viabilities and $p = (p_j)$ the column vector of frequencies. Furthermore define the diagonal matrix $P = (p_j \delta_{jk})$. Always a^* (transpose) is the row vector corresponding to the column vector a, in particular $e^* = (1, \ldots, 1)$. Then a^*b is a scalar product, ba^* is a dyad, and a^*Fa is a quadratic form. With this notation equations (1) read

$$(2) \qquad \dot{p} = PFp - p^*Fp \cdot p.$$

We consider the population distributed in a bounded spatial domain Ω of $\mathbf{R}^m$ with smooth (say C^3) boundary and describe migration and reproductive interaction of neighboring individuals by a diffusion term

$$(3) \qquad p_t = PFp - p^*Fp \cdot p + \Delta p.$$

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