ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 10, Number 3, Summer 1980

A T_B SPACE WHICH IS NOT KATETOV T_B

WILLIAM G. FLEISSNER

In 1943, E. Hewitt [1] proved the beautiful theorem that a compact Hausdorff space is minimal Hausdorff and maximal compact. Restating this result in more detail, if (X, τ) is a compact Hausdorff space and (X, τ') and (X, τ'') are spaces $\tau' \subsetneq \tau \subsetneq \tau''$, then (X, τ') is not Hausdorff, and (X, τ'') is not compact. The converses to this theorem are appealing but false. There are noncompact minmal Hausdorff spaces [2] and non Hausdorff maximal compact spaces [2].

A compact space is maximal compact if every compact set is closed [3]. Let us call spaces in which all compact sets are closed T_B spaces, as this notion can be thought of as a separation axiom between T_1 and T_2. They are also called *KC* spaces. R. Larson [4] asked whether a space is maximal compact iff it is minimal T_B. A related question is whether every T_B topology is Katetov T_B, that is whether every T_B topology contains a minimal T_B topology. The author wishes to thank Douglas Cameron for bringing these questions to his attention. In this paper we construct a T_B not Katetov T_B tpace.

The point set of all spaces in this paper will be the countable ordinals. To avoid ambiguity, we will refer to the first uncountable ordinal (and cardinal) as ω_1, and to the point set of the spaces as Ω. A typical point of Ω will be x_α, where $\alpha < \omega_1$. The point set $\{x_\beta : \beta < \alpha\}$ will be called $P(\alpha)$, the predecessors of α; and the point set $\{x_\beta : \beta > \alpha\}$ will be called $S(\alpha)$, the successors of α. The usual topology on Ω, generated by $\{P(\alpha): \alpha < \omega_1\} \cup \{S(\alpha) : \alpha < \omega_1\}$ will be called $\mathscr{u}$. The cardinality of a set S will be denoted $|S|$.

LEMMA 1. *If $\tau' \subset \tau$ and K is τ compact, K is τ' compact.*

LEMMA 2. *A compact T_B space is a minimal T_B space.*

If $S \subset \Omega$, we denote the subspace of (Ω, τ) with point set S by $(S, \tau|S)$. Equivalently, $\tau|S = \{U \cap S: U \in \tau\}$. We say that τ, τ' agree on countable sets if for all $S \subset \Omega$ with $|S| \leqq \omega$,

$$(S, \tau|S) = (S, \tau'|S).$$

LEMMA 3. *Suppose $\tau \subset \mathscr{u}$ and (Ω, τ) is T_B. Then $\tau, \mathscr{u}$ agree on countable sets.*

Received by the editors on July 29, 1977.