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COMPOSITION OPERATORS ON A SPACE OF LIPSCHITZ FUNCTIONS

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ABSTRACT. For $0 < \alpha \leqq 1$, let $\mathrm{Lip}(\alpha)$ denote the space of functions f which are analytic on the open unit disk, continuous on the closed unit disk, and whose boundary values satisfy a Lipschitz condition of order $\alpha : |f(z) - f(w)| \leqq K|z - w|^\alpha$, for $|z| = |w| = 1$. For $0 < \alpha < 1$, let $\mathrm{lip}(\alpha)$ denote the space of functions f in $\mathrm{Lip}(\alpha)$ such that $|f(z) - f(w)| = o(|z - w|^\alpha)$, as $w \to z$, $|z| = |w| = 1$. We prove that a function φ in $\mathrm{Lip}(\alpha)$ (resp., $\mathrm{lip}(\alpha)$), with $|\varphi(z)| \leqq 1$ for $|z| \leqq 1$, induces a composition operator on $\mathrm{Lip}(\alpha)$ (resp., $\mathrm{lip}(\alpha)$) if and only if there exists a finite number M and a number $r < 1$ such that $|\varphi(z)| \geqq r$ implies $|\varphi'(z)| \leqq M$. We also prove that a composition operator C_φ on either $\mathrm{Lip}(\alpha)$ or $\mathrm{lip}(\alpha)$ is compact if and only if for each $\epsilon > 0$ there exists an $r < 1$ such that $|\varphi(z)| \geqq r$ implies $|\varphi'(z)| \leqq \epsilon$.

1. **Introduction.** We shall denote the unit disk $\{|z| < 1\}$ by $\mathbf{U}$. For $0 < \alpha \leqq 1$, we let $\mathrm{Lip}(\alpha)$ denote the space of functions f which are analytic in $\mathbf{U}$, continuous on $\mathbf{U}^-$ (the closure of $\mathbf{U}$), and whose boundary values satisfy a Lipschitz condition of order α:

$$\frac{|f(z) - f(w)|}{|z - w|^\alpha} = o(1), \qquad |z| = |w| = 1.$$

For $0 < \alpha < 1$, we let $\mathrm{lip}(\alpha)$ denote those functions f in $\mathrm{Lip}(\alpha)$ for which

$$\frac{|f(z) - f(w)|}{|z - w|^\alpha} = o(1) \qquad \text{as } w \to z,\ |z| = |w| = 1.$$

Each of the spaces $\mathrm{Lip}(\alpha)$ and $\mathrm{lip}(\alpha)$ is a Banach algebra when the norm of an element is defined as

$$\|f\|_\alpha = \|f\|_\infty + \sup_{\substack{z \neq w \\ |z|=|w|=1}} \frac{|f(z) - f(w)|}{|z - w|^\alpha},$$

where $\|f\|_\infty = \sup|f(z)|$ $(|z| < 1)$.

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