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ON THE NONNEGATIVITY OF NORMAL HILBERT COEFFICIENTS OF TWO IDEALS

SHREEDEVI K. MASUTI AND J.K. VERMA

Dedicated to Professor Jürgen Herzog on the occasion of his 70th birthday

ABSTRACT. Let $(R, \mathfrak{m})$ be an analytically unramified local ring of dimension $d \geq 1$, and let I, J be $\mathfrak{m}$-primary ideals. Let $\overline{e}_{(i,j)}(I, J)$ be the coefficient of $(-1)^{d-(i+j)}\binom{x+i-1}{i}\binom{y+j-1}{j}$ of the normal Hilbert polynomial of I and J. In this paper we prove that $\overline{e}_{(i,j)}(I, J)$ are nonnegative for $i + j \geq d - 3$ in Cohen-Macaulay local rings. We also prove that, if $i + j = d-1$, then $\overline{e}_{(i,j)}(I, J)$ are nonnegative in unmixed local rings.

1. Introduction. Let R be a commutative ring and I an ideal of R. We say that $x \in R$ is integral over I if x satisfies

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

for some $a_i \in I^i, i = 1, 2, \ldots, n$. The set $\overline{I}$ of elements that are integral over I is an ideal, called the *integral closure* of I. A Noetherian local ring $(R, \mathfrak{m})$ is called *analytically unramified* if its $\mathfrak{m}$-adic completion is reduced. For an $\mathfrak{m}$-primary ideal I in an analytically unramified local ring R of dimension d, there exist uniquely determined integers $\overline{e}_0(I), \ldots, \overline{e}_d(I)$ such that, for large n,

$$\lambda(R/\overline{I^{n+1}}) = \overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d\overline{e}_d(I),$$

where λ denotes length [**10**, Theorem 1.4] and [**11**, Theorem 1.1]. Bhattacharya [**1**, Theorem 8] showed that, for $\mathfrak{m}$-primary ideals I and J in a Noetherian local ring $(R, \mathfrak{m})$ of dimension d, there exist integers $e_{(i,j)}(I, J)$ such that, for large r, s,

$$\lambda(R/I^rJ^s) = \sum_{i+j\leq d} (-1)^{d-(i+j)}e_{(i,j)}(I, J)\binom{r+i-1}{i}\binom{s+j-1}{j}.$$

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