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Subrings of a polynomial ring of one variable

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The following problem was communicated to the writer by Dr. A. Zaks of the University of Oregon:

We consider the polynomial ring $A[X]$ of one variable X over a normal domain A. Give a criterion for a ring R to coincide with $A[X]\cap K$ with a suitable field K containing A.

In this article, we give an answer as follows:

Theorem 1. *Such an R is characterized by the property that there is a polynomial f which belongs to $XA[X]$ (i.e., the constant term of f is zero) such that R is generated by $S_i=\{g\in A[X]\mid \exists a, b\in A, a\neq 0, ag=bf^i\}$ $(i=1, 2, \cdots)$.*

As for the proof, if $R=A$, then f is zero, and we assume that $R\neq A$. On the other hand, let k and L be the fields of quotients of A and R, respectively. Then we may assume that $K=L$. First we prove a lemma:

Lemma. 2 *Assume that A is a valuation ring of k and that $f=c_1X^n+c_2X^{n-1}+\cdots+c_nX$ is a polynomial over A such that some of the coefficients c_i are units in A. Then a polynomial $h=e_0+e_1f+\cdots+e_sf^s$, in f with coefficients e_i in k, is in $A[X]$ if and only if all e_i are in A.*

Proof. The if part is obvious, and we want to prove the only if part. Assume that $h\in A[X]$. $e_0=h(0)$, and therefore $e_0\in A$. Then $f(e_1+\cdots+e_sf^{s-1})\in A[X]$. Since f is a primitive polynomial, we see that $e_1+\cdots+e_sf^{s-1}\in A[X]$. Thus we prove the assertion by induction on s. QED

The if part of Theorem 1 follows from the following result:

Proposition 3. *Under the assumption at the beginning, if $f\in XA[X]$, then $A[X]\cap k(f)$ is the ring generated by S_i $(i=1, 2, \cdots)$ over A.*

Proof. It is obvious that all the S_i are contained in $A[X]\cap k(f)$. Conversely, let h be an arbitrary element of $A[X]\cap k(f)$. We may assume that $f=c_1X^n+c_2X^{n-1}+\cdots+c_nX$, $c_i\in A$, $c_1\neq 0$. Then X is integral over $A[f, c_1^{-1}]$ and therefore $A[X]\cap k(f)\subseteq A[f, c_1^{-1}]$. This shows that $h=e_0+e_1f+\cdots+e_sf^s$ with e_i in $A[c_1^{-1}]\subseteq k$. Since A is normal, A is the intersection of valu-