J. Math. Kyoto Univ. (JMKYAZ)
23-2 (1983) 357-367

Dirichlet finite harmonic differentials with integral periods on arbitrary Riemann surfaces

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(Received March 19, 1982)

Introduction

In this paper we consider Dirichlet finite harmonic differentials with integral periods on arbitrary Riemann surfaces. From such a differential σ on an arbitrarily given Riemann surface R, we can construct a mapping $u_\sigma(p)$ (, for the definition, see §1,) from R into $S^1=\{|z|=1\}$, and we can take $u_\sigma^{-1}(t)$ as a "level set" of σ for every $t\in S^1$. Such a mapping can be extended continuously onto the Royden's compactification R^* of R. Now Theorem 1 in §1 states that for almost all t in $S^1-u_\sigma(\Delta)$ the set $u_\sigma^{-1}(t)$ consists only of (at most countable number of) simple closed (, hence compact) curves in R, where Δ is the harmonic boundary of R^*. In particular, if $u_\sigma(\Delta)$ is a set of linear measure zero on S^1, then the holomorphic quadratic differential $(-{}^*\sigma+\sqrt{-1}\cdot 1\sigma)^2$ has closed trajectories (in the sense of K. Strebel).

Next Theorem 2 states that if t_1 and t_2 are contained in the same component of $S^1-u_\sigma(\Delta)$, then the "level sets" $u_\sigma^{-1}(t_1)$ and $u_\sigma^{-1}(t_2)$ have same length with respect to the metric naturally induced by σ (, or equivalently, ${}^*\sigma$ has same periods along $u_\sigma^{-1}(t_1)$ and $u_\sigma^{-1}(t_2)$ with suitable orientations).

Definitions and main theorems are stated in §1, and the applications are made to basic differentials and functions such as reproducing differentials for 1-cycles, Green's functions and harmonic measures in §2. Proofs of main theorems are given in §3, and examples are provided in §4.

§1. Definitions and main results

Let R be an arbitrary Riemann surface and $\Gamma_h(R)$ be the Hilbert space of square integrable *real* harmonic differentials on R. We say that a differential σ in $\Gamma_h(R)$ has *integral periods* if $\int_c \sigma$ is an integer for every 1-cycle c on R, and set

$$\Gamma_{hI}(R)=\{\sigma\in\Gamma_h(R):\ \sigma \text{ has integral periods}\}.$$

Here note that $\Gamma_{he}(R)$ is clearly contained in $\Gamma_{hI}(R)$. For every $\sigma\in\Gamma_{hI}(R)$ and arbitrarily fixed point $p_0\in R$ and real constant a_0,