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Towards a generalization of Mumford's Theorem

By

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§1. Introduction

In this paper, we examine a theorem of Mumford, based on a technique of Severi, on the role of regular 2-forms in determining the "size" of the Chow group of 0-cycles on a smooth, complex, projective algebraic surface (cf [6]). Let X be a smooth, complex projective algebraic variety of dimension n, $CH_k(X) = CH^{n-k}(X)$ = Chow group of algebraic cycles of dimension k (codim $n-k$) modulo rational equivalence, and $A_k(X) = \{v \in CH_k(X) | v$ is algebraically equivalent to zero$\}$. On rational cohomology (of X) there is equipped an arithmetic filtration, for which to quote in [2], "... embodies deep arithmetic properties of [the scheme] X". Our purpose is to apply Mumford's theorem to exhibit a connection between graded pieces of this filtration and the "size" of $A_k(X)$ for $k \geq 0$. According to the statement of the main theorem below, when $k = 0$, we arrive at the conclusion of Mumford's theorem for the case $n = 2$, and Roitman's generalization (see [8]) for $n \geq 2$, namely $h^{p,0}(X) \neq 0$ for some $p \geq 2$ implies $A_0(X)$ "infinite dimensional".

We begin with the setting of this paper by recalling two filtrations (see also [2]). $F_a^k(i)$ = Gysin images of $H^{i-2q}(Y, \mathbf{Q})$ in $H^i(X, \mathbf{Q})$, where Y = desingularization of T and where $T \subset X$ is a subvariety of codimension $q \geq k$; $F_H^k(i)$ = largest subHodge structure in $\{F^k H^i(X, \mathbf{C})\} \cap H^i(X, \mathbf{Q})$. There is the well known inclusion $F_a^k(i) \subset F_H^k(i)$, conjectured to be an equality ([Grothendieck amended] Hodge conjecture), and corresponding graded morphisms $T_{k,i}$: $Gr_a^k(i) = F_a^k(i)/F_a^{k+1}(i) \rightarrow F_H^k(i)/F_H^{k+1}(i) = Gr_H^k(i)$, which again translate to isomorphisms if one assumes the Hodge conjecture to be true. Now let V be a smooth projective variety, j: $W \hookrightarrow V$ a smooth hyperplane section, and recall the weak Lefschetz theorem, namely j^*: $H^i(V) \rightarrow H^i(W)$ injective (resp. isomorphism) for $i = \dim W$ (resp. $i < \dim W$). Let$(j^*)^{-1}$ be the left inverse to j^* (as introduced in [4]). Our main assumption is a standard conjecture of Lefschetz type:

$A(*)$: $(j^*)^{-1}$ is algebraic, i.e. induced by an algebraic cycle with rational coeffi-

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