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On a probabilistic properties of Takagi's function

By

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Let

$$\Phi(x)=\begin{cases} 2x & \text{if } 0\leqq x\leqq 1/2\,, \\ 2-2x & \text{if } 1/2\leqq x\leqq 1\,, \end{cases}$$

and define

$$f(x)=\sum_{n=1}^{\infty} 2^{-n}\Phi^{(n)}(x),$$

where $\Phi^{(n)}(x)=\Phi(\Phi^{(n-1)}(x))$ such that $\Phi^{(0)}(x)=x$. This function $f(x)$ is known as Takagi's nowhere differentiable continuous function [1]. It should be noted that much later van der Waerden rediscovered the Takagi function (see [2] or [3]). Since then a number of scientists have studied this function from various points of views [4, 5].

The aim of this paper is to investigate another properties of Takagi's function. In particular it is proved that the local modulus of continuity of the function $f(x)$, after appropriate normalization is asymptotically normal (Theorem 1). In a previous paper it had been shown the same for Weierstrass' function [8]. Theorem 2 due to N. Kôno [5] and it seems to me that this another proof is not more difficult than in [5].

Theorem 1. *Let $f(x)$ be the Takagi function then*

$$\lim_{h\downarrow 0} mes\left\{x:\ x\in(0,\,1)\ \frac{f(x+h)-f(x)}{h\sqrt{\log_2(1/h)}}<y\right\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-u^2/2}du\,,$$

where $x+h$ here and below is defined as the sum modulo 1 *and* $h>0$.

Proof. Let us consider the points x and $x+h$ by binary series

$$x=\sum_{k=1}^{\infty}\varepsilon_k 2^{-k},\qquad x+h=\sum_{k=1}^{\infty}\varepsilon'_k 2^{-k} \tag{1}$$

and let $\{X_n(x)\}$ be Rademacher system of functions that is $X_n(x)=1-2\varepsilon_n$.

Assume that $h\leqq 1/2$. Then there exist $m=m_h$ such that

$$\frac{1}{2^{m+1}}<h\leqq\frac{1}{2^m} \tag{2}$$

and denote by k_0 random variable $k_0=k_0(x,\,h)=\{\max k:\ \varepsilon_1=\varepsilon'_1,\,\cdots,\,\varepsilon_k=\varepsilon'_k\}$ (it depends on random variable $X_n(x)$). Using now (1) we can write

$$P(k_0=0)=2h,\qquad P(k_0=l)=h2^l\qquad 1\leqq l\leqq m-1,$$

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