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A NOTE ON THE VOLUME OF A RANDOM POLYTOPE IN A TETRAHEDRON

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1. Introduction

The following problem in integral geometry was presented by Klee [5] in 1969:

From a simplex of unit volume in Euclidean d-space $d + 1$ points are chosen independently and uniformly at random. What is the expected volume $\mathscr{V}(d)$ of their convex hull?

Let us consider the more general question of determining the expected volume $\mathscr{V}(d, n)$ of the convex hull of $n \geqq d + 1$ points chosen independently and uniformly from the given d-simplex. If $d = 1$, almost trivial calculations yield

$$\mathscr{V}(1, n) = 1 - \frac{2}{n+1}.$$

In the case $d = 2$, the expected volume of the convex hull of n points chosen independently and uniformly from an arbitary convex polygon was derived in [1]; especially,

$$\mathscr{V}(2, n) = 1 - \frac{2}{n+1} \sum_{k=1}^{n} \frac{1}{k}.$$

For $d = 3$, it is only known (cf. [2]) that

$$\mathscr{V}(3,5) = \frac{5}{2}\mathscr{V}(3,4).$$

In the following we give a representation of $\mathscr{V}(3, n)$ by a threefold definite integral, whence we deduce that

$$1 - \mathscr{V}(3, n) \sim \frac{3}{4}\frac{\log^2 n}{n}$$

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