SOME PROBLEMS ON THE PRIME FACTORS OF CONSECUTIVE INTEGERS

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In this note we are going to discuss some elementary questions on consecutive integers. Though the problems are all quite elementary we are very far from being able to solve them.

For n a positive integer and k a non-negative integer let

$$v(n;\, k) = \textstyle\sum_{p \mid n+k,\ p>k} 1.$$

In other words $v(n; k)$ denotes the number of prime factors of $n + k$ which do not divide $n + i$ for $0 \le i < k$. Put

$$v_0(n) = \max_{0 \le k < \infty} v(n;\, k).$$

One would expect that $v_0(n) \to \infty$ as $n \to \infty$ but we are very far from being able to prove this. We can only show that $v_0(n) > 1$ for $n \ge 17$. In fact we can show the following result: $v_0(n) > 1$ for all n except $n = 1, 2, 3, 4, 7, 8, 16$.

It is easy to see that $v_0(n) = 1$ for the above values of n. In general if $k > 1$ then $v(n; k) \le 1$ for $k^2 + 3k > n - 3$. In fact for $k \ge n$ we have $v(n; k) = 1$ if and only if $n + k$ is a prime.

Clearly $v_0(n) = 1$ implies $n = p^\alpha$. Assume first p odd. $v_0(n) = 1$ implies $\nu(p^\alpha + 1) = 1$ or $p^\alpha + 1 = 2^\beta$. (Here $\nu(m)$ denotes the number of distinct prime factors of m.) $p = 3$ is impossible for $n > 3$ since $3^\alpha + 1 = 2^\beta$ is impossible for $\alpha > 1$. But then $n + 2 \equiv 0 \pmod 3$ and $n + 2 = 2^\beta + 1 = 3^\gamma$, but this is also known to be impossible for $\beta > 3$ i.e., for $n > 7$. If n is even then $n = 2^\alpha$, $2^\alpha + 1 = g^\beta$, $g \ne 3$ since $\alpha > 3$. Thus $2^\alpha + 2 = 2 \cdot 3^\gamma$ which is impossible since $\alpha > 4$. Thus our result is proved.

Put

$$v_l(n) = \max_{l \le k < \infty} v(n;\, k).$$

It seems certain that

$$\lim_{n=\infty} v_l(n) = \infty$$

for every l, but unfortunately we have not even been able to prove that $v_1(n) = 1$ has only a finite number of solutions, though this certainly must be true. Probably the greatest n for which $v_1(n) = 1$ is $n = 330$, but we have no method of proving this. In fact, $v_1(n) = 1$ for $n =$ 1–4, 6–8, 10, 12, 15, 16, 18, 22, 24, 26, 30, 36, 42, 46, 48, 60, 70, 78, 80, 96, 120, 190, 222, 330, and for no other values of $n < 2500$.

A slight modification of this problem might be more amenable to attack.

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