HIGHER ORDER WHITEHEAD PRODUCTS AND POSTNIKOV SYSTEMS

BY

GERALD J. PORTER[1]

In an arcwise connected CW-complex, higher order Whitehead products are determined, as are all homotopy operations, by the Postnikov invariants of the space. This fact has been used implicitly in [3] to prove that certain products are non-zero. In this note we calculate this relationship explicitly. This is done by relating Whitehead products and classical obstruction theory. Let $P^n(C)$ denote n-dimensional complex projective space. As an application we show that if $\iota \in \pi_2(P^n(C))$ is a generator, then the set of $(n+1)^{st}$ order Whitehead products $[\iota, \cdots, \iota]$ equals $(n+1)!\ \sigma$, where σ is a generator of $\pi_{2n+1}(P^n(C))$.

The author would like to thank M. Arkowitz for raising the question about $P^n(C)$ which led to this note.

Let T denote the subset of the cartesian product, $\times_{i=1}^{k} S^{n_i}$, consisting of those points with at least one coordinate at a base point. We assume throughout that $n_i > 1$, all i, and $k \geq 2$. Choose a generator $\mu \in H_N(\times S^{n_i}; Z)$, where $N = \sum n_i$. Given a map $g : T \to X$, the k^{th} order Whitehead product, $W(g) \in \pi_{N-1}(X)$, is defined by $W(g) = g_* \partial H j_*(\mu)$, where j_* is induced by the inclusion,

$$j : (\times S^{n_i}, *) \to (\times S^{n_i}, T),$$

H is the Hurewicz homomorphism, and ∂ is the boundary in the homotopy sequence of the pair $(\times S^{n_i}, T)$. These products were defined and studied in [2]. It was shown there that g can be extended to the cartesian product if and only if $W(g) = 0$.

On the other hand classical obstruction theory yields an element,

$$o(g) \in H^N(\times S^{n_i}, T; \pi_{N-1}(X)),$$

such that g can be extended if and only if $o(g) = 0$. (We use here the fact that the $(N-2)$ skeleton of $\times S^{n_i}$ equals T.)

Let $\langle\ ,\ \rangle$ denote the Kronecker pairing

$$H^N(\times S^{n_i}; \pi_{N-1}(X)) \otimes H_N(\times S^{n_i}; Z) \to \pi_{N-1}(X).$$

The following lemma is then evident.

LEMMA 1. $\langle j^* o(g), \mu \rangle = W(g)$.

Given a fibre space (E, p, B) with fibre F and a map $g : T \to E$ such that pg can be extended to $h : \times S^{n_i} \to B$, the usual obstruction theory for cross-

Received May 26, 1966.

[1] The author was an Office of Naval Research postdoctoral research fellow during part of the period in which this note was written.