MEAN GROWTH AND COEFFICIENTS OF H^p FUNCTIONS[1]

BY

P. L. DUREN AND G. D. TAYLOR

Let $f(z)$ be analytic in the unit disk $|z| < 1$, and let

$$M_p(r,f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p}, \qquad 0 < p < \infty;$$

$$M_\infty(r,f) = \max_{|z|=r} |f(z)|.$$

The function f is said to belong to the class H^p $(0 < p \leq \infty)$ if $M_p(r,f)$ is bounded for $0 \leq r < 1$. Hardy and Littlewood [4], [5] proved that $f \in H^p$ implies

$$M_q(r,f) = o((1-r)^{1/q-1/p}), \quad 0 < p < q \leq \infty,$$

and they pointed out that the exponent $(1/q - 1/p)$ is best possible. In the present paper, we show that the Hardy-Littlewood estimate is best possible in a stronger sense, and we apply this result to prove that several known theorems on the Taylor coefficients of H^p functions are also best possible.

THEOREM 1. *Let $0 < p < q \leq \infty$, and let $\phi(r)$ be positive and non-increasing on $0 \leq r < 1$, with $\phi(r) \to 0$ as $r \to 1$. Then there exists a function $f \in H^p$ such that*

$$M_q(r,f) \neq O(\phi(r)(1-r)^{1/q-1/p}).$$

For $q = \infty$, this theorem was obtained in [6]. The more general result is now deduced from this special case. We shall need the following elementary lemma (see [2, Kap. IX, §5]).

LEMMA. *Let $1 < p < \infty$, and let $\rho = (1+r)/2$, where $0 < r < 1$. Then as $r \to 1$,*

$$\int_0^{2\pi} |\rho e^{it} - r|^{-p} \, dt = O((1-r)^{1-p}).$$

Proof of Theorem 1. Let $f \in H^p$, $p < q$, and suppose first that $1 < q < \infty$. If $\rho = (1+r)/2$, we have

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad z = re^{i\theta}.$$

Thus, by Hölder's inequality and the lemma,

$$M_\infty(r,f) \leq C(1-r)^{-1/q} M_q(\rho, f).$$

From this it is clear that the theorem for $1 < q < \infty$ follows from the case

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