ON FOURS GROUPS

BY

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In a recent paper [2], Ernest Shult has discovered remarkable necessary and sufficient conditions for a conjugacy class of involutions of a group to be the set of non-identity elements of a subgroup of order greater than two. Even more striking is the fact that this result is a consequence of a theorem characterizing the symplectic groups over two-element fields. In trying to give a direct proof of this corollary we have, in fact, come up with a stronger result, namely:

THEOREM. *If V is a fours subgroup of a group G and V intersects $O_2(G)$ trivially, then there is an involution of G, conjugate to an element of V, which commutes with no involution of V.*

It turns out that this strange theorem can even be used in studying doubly transitive groups, in places where Shult's result is not strong enough [3]. We shall now proceed by first proving the theorem and then stating and deriving Shult's result from it. All our notation is standard [1].

Since $V \cap O_2(G) = 1$, Baer's theorem [1, p. 105] yields that each involution of V has a conjugate together with which it generates a subgroup of order not a power of two. However, this subgroup is dihedral as it is generated by two involutions. Thus, it follows that each element of V inverts a non-identity element of odd order of G.

Let $a \in V^{\#}$ and choose such an element x. If a^x centralizes no element of $V^{\#}$ we are done; thus we may assume a^x does centralize an element b of $V^{\#}$. But $a^x = x^{-1}ax = ax^2$ and V is abelian so x^2 centralizes b. In particular, $b \neq a$. Moreover, x centralizes b since x is a power of x^2 as x has odd order.

Similarly, b inverts a non-identity element y of odd order and we may assume that y centralizes an element of $V^{\#}$ other than b. If y centralizes a then we have the following symmetrical relations:

$$x^a = x^{-1}, \quad x^b = x, \quad y^a = y, \quad y^b = y^{-1}.$$

On the other hand, suppose that y centralizes ab, the other element of $V^{\#}$. We set $a' = ab$ so a' inverts x, as x does and y centralizes x. As a' is assumed to centralize y, we see that if we replace a by a' then we again have the above symmetrical relations. Hence, we shall assume this is done.

There are now two possibilities to consider: x and y commute or they do not. First, we claim that if x and y do commute then $(ab)^{xy}$ is the desired

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