A NOTE ON MODULES WITH WAISTS

BY

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In [2], a class of indecomposable modules was introduced, namely modules which contain a waist. Recall that a module M contains a waist if there is a proper nonzero submodule N of M which either contains or is contained in each submodule of M. The purpose of this note is to extend Theorems 2.2 and 2.4 of [2] to arbitrary Artin algebras, not just those with radical square zero. Recall also that an Artin algebra Λ is an Artin ring which is a finitely generated module over its center, C, which is a commutative Artin ring. The most important class of Artin algebras is the class of rings which are finite dimensional over a field. If Λ is an Artin algebra with center C then there is a duality, D, between the categories of left and right finitely generated Λ-modules. This duality is given by $D(X) = \operatorname{Hom}_C(X, E)$ where X is finitely generated left or right Λ-module and E is the C-injective envelope of $C/\operatorname{rad}(C)$. Finally, we also correct a misprint in [2].

We fix some notation for this paper. Unless otherwise stated, all modules will be finitely generated left modules. Let Λ be an Artin algebra with radical $\mathbf{r}$ and let X be a Λ-module. Then the top of X, denoted top (X), is $X/\mathbf{r}X$. The Loewy length of X, denoted $ll(X)$, is smallest integer n such that $\mathbf{r}^n X = 0$. Finally, the lower Loewy series for X,

$$0 \subset_{\neq} S^0(X) \subset S^1(X) \subset \cdots \subset S^t(X) = X$$

is defined by $S^0(X) = \operatorname{soc}(X)$, the socle of X, and

$$S^i(X) = \pi_i^{-1}(\operatorname{soc}(X/S^{i-1}(X)))$$

where $\pi_i\colon X \to X/S^{i-1}(X)$ is the canonical surjection.

THEOREM 1. *Let Λ be an Artin algebra. If Λ is of finite representation type then every module containing a waist has either a simple top or simple socle.*

Proof. Assume that Λ is of finite representation type and let X be a module containing a waist. By [2, Proposition 1.4], X is finitely generated. Assume that both the top of X and the socle of X are not simple.

By [3, Theorem 2.2], it follows that $ll(X) \geq 3$. We proceed by induction on $ll(X)$. We first show that top $(\mathbf{r}X)$ is simple. If top $(\mathbf{r}X)$ is not simple then either $\mathbf{r}X$ or $X/\mathbf{r}^2X$ has a waist contradicting the induction hypothesis. By [2, Proposition 1.3] $\mathbf{r}X$ is a waist in X.

Received February 6, 1976.

[1] This research was partially supported by a grant from the National Science Foundation.