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FUNCTION SPACE COMPLETIONS

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Many problems in analysis require the introduction of a linear space G of functions. The next step is to equip G with a norm which is appropriate for the specific problem. This in turn leads to the necessity of constructing a complete space $\hat{G}$ (Banach space) by adjoining functions to G. We assume that the domain and range spaces remain fixed. Ordinarily the norm of a function in G will be defined in terms of the values the function assumes at points in the domain. If the same statement cannot be made for functions in $\hat{G}$, then it is unlikely that $\hat{G}$ will be of much use. The purpose of this paper is to overcome this difficulty.

We begin by describing the relationship between the norm of a function in G and the values the function assumes at points in the domain. This can always be done. The result appears as Theorem 1 of [4] and again in this paper as Corollary 5.4 to Theorem 5.3. The relationship between norm and functional values is formulated in such a way that it suggests which functions should be added to G to form the completion. The resulting normed space H has the property we seek. The norm of a function in H is defined in terms of its values at points in the domain and, moreover, the definition is derived from the manner in which norms of functions in G are related to their functional values. On the other hand, it is not always possible to obtain a completion by adjoining functions. This fact is illustrated by Example 3.3.

Our approach to the problem of obtaining function space completions parallels the treatment of Grothendieck's completion theorem given in [8]. We also view the problem as having two parts: First we obtain function spaces in which the original space is dense. Secondly, we look for complete spaces among these. Our key tool is Theorem 2.3 which generalizes the first part of Grothendieck's theorem and characterizes the additional functions and their norms. Grothendieck's theorem is only applicable to the special case in which the domain of the functions in G is the continuous dual G'. Section 3 gives several sufficient conditions for the resulting function space to be complete. The theorems are then applied to give new characterizations of L_p spaces $(1 \leq p < \infty)$ as completions of the space of continuous functions. The characterizations are totally different from any we have seen in the literature.

More must be said about the relationship of the norm to the point values of the functions. It is unrealistic to think that a norm mysteriously appears on a

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