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THE DIMENSIONS OF PERIODIC MODULES OVER MODULAR GROUP ALGEBRAS

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1. Introduction

Let G be a finite group and let K be a field of characteristic $p>0$. We shall assume that all KG-modules are finitely generated and hence have finite K-dimensions. A KG-module M is periodic if there exists an exact sequence

$$(1.1) \qquad 0 \to M \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of KG-modules such that $P_0, \ldots, P_{n-1}$ are projective. The period of M is the least length n of any such sequence.

We prove in this paper that if G is an abelian p-group and if M is an indecomposable periodic KG-module, then there is a subgroup H of G such that G/H is cyclic and the restriction of M to a KH-module is free. This implies that the period of M is at most 2. For any finite group G, the dimension of a periodic KG-module is divisible by p^{r-1} where r is the p-rank of G. That is, the maximal elementary abelian p-subgroup of G has order p^r. These results answer some questions raised by Alperin in [1].

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2. Notation and preliminaries

Throughout this paper G denotes a finite group and K is a field of characteristic $p>0$. The radical of KG is denoted $\operatorname{Rad} KG$. If H is a subgroup of G and M is a KG-module, then M_H is the restriction of M to a KH-module. The socle of M, $\operatorname{Soc}(M)$, is the sum of the minimal submodules of M. If G is a p-group, then

$$\operatorname{Soc}(M)=\{m \in M \mid xm=m \quad \text{for all} \quad x \in G\}.$$

Let $\tilde{H}=\sum_{h \in H} h \in KG$, and let $1(G)=K$ denote the trivial one-dimensional KG-module. The symbol $U(KG)$ denotes the group of units in KG.

For any KG-module M there exists a projective module F and an epimorphism $\varphi\colon F \to M$. Let $\Omega(M)$ be the direct sum of the nonprojective

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