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GROWTH PROBLEMS FOR A CLASS OF ENTIRE FUNCTIONS VIA SINGULAR INTEGRAL ESTIMATES

BY

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Let f be an entire function with zeros $\{z_n\}$, and let

$$M(r,f) = \max_\theta |f(re^{i\theta})|, \qquad L(r,f) = \min_\theta |f(re^{i\theta})|,$$

$$n(r) = n(r, 0; f) = \sum_{|z_n| \le r} 1,$$

$$\rho = \limsup_{r\to\infty} \frac{\log \log M(r,f)}{\log r}.$$

We consider a problem motivated by a classical theorem of Pólya and Valiron; this gives lower bounds for

$$c(f) = \limsup_{r\to\infty} \frac{n(r)}{\log M(r,f)} \tag{1}$$

for functions of finite nonintegral order ρ. Let

$$C(\rho) = \inf\{c(f) : f \text{ of order } \rho\}.$$

Then Pólya [6] and Valiron [9] [10] proved, independently, that

$$C(\rho) = \frac{1}{\pi} \sin \pi\rho \quad (0 \le \rho \le 1), \tag{2}$$

$$\frac{1}{\pi} |\sin \pi\rho| \ge C(\rho) \ge \frac{|\sin \pi\rho|}{A_0\{\log \rho + 1\}|\sin \pi\rho| + \pi} \quad (1 < \rho < \infty), \tag{3}$$

for an absolute constant A_0. The upper estimate in (3) comes from the Lindelöf functions

$$f_\rho(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^\sigma}\right) \exp\left\{\sum_{k=1}^{q} \frac{1}{k}\left(\frac{z}{-n^\sigma}\right)^k\right\} \quad (\sigma = \rho^{-1}, q = [\rho]),$$

for which $n(r) \sim r^\rho$, $\log M(r,f) \sim \pi |\csc \pi\rho| r^\rho$ when ρ is nonintegral and $r \to \infty$. We conjecture that these f_ρ, having all zeros regularly distributed on a single ray $\arg z = \text{constant}$, are extremal for this problem, i.e. that $C(\rho) = \pi^{-1} |\sin \pi\rho|$. But not even the order of magnitude of $C(\rho)$ is known, for ρ large.

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